

DISCRETE GROUP ACTIONS ON SPACETIMES: CAUSALITY CONDITIONS AND THE CAUSAL BOUNDARY

STEVEN G. HARRIS

ABSTRACT. Suppose a spacetime M is a quotient of a spacetime V by a discrete group of isometries. It is shown how causality conditions in the two spacetimes are related, and how can one learn about the future causal boundary on M by studying structures in V . The relations between the two are particularly simple (the boundary of the quotient is the quotient of the boundary) if both V and M have spacelike future boundaries and if it is known that the quotient of the future completion of V is past-distinguishing. (That last assumption is automatic in the case of M being multi-warped.)

0: INTRODUCTION

Symmetry conditions on spacetimes are often employed to simplify various considerations and make important features stand out. Such symmetries are often expressed by the discrete action of a group of isometries; an example is the fundamental group acting on the universal cover of a space.

The impetus for this paper is to isolate and understand the effect of these kinds of constructions on the causal structures of spacetimes, particularly in such properties as strong causality and the causal boundary of Geroch, Kronheimer, and Penrose [GKP], [HE] (as topologized in [H2]). To this effect, we will look at the situation of a principle (or regular) covering projection of spacetimes $\pi : V \rightarrow M$ given by a discrete group G of isometries acting freely and properly discontinuously on V : $M = V/G$ and $\pi(x) = G \cdot x$ is the orbit of x under G .

This context is inspired by the thought of starting with a spacetime M and letting V be \widetilde{M} , the universal covering space of M , with G the fundamental group of M , $\pi_1(M)$. But the results apply just as well for $V = \widetilde{M}/H$ for H any normal subgroup of $\pi_1(M)$, with $G = \pi_1(M)/H$; and some of the results also apply for a non-principle (non-regular) cover, *i.e.*, with $V = \widetilde{M}/H$ for H a non-normal subgroup of G and $\pi : V \rightarrow M$ a covering projection not given by a group action.

An important application of the results of this paper will be an investigation (in a subsequent paper) of the causal boundary of any strongly causal static-complete spacetime (static with the timelike Killing field being complete). Any such spacetime has a standard static spacetime as its universal covering space, and the causal boundary of standard static spacetimes was given in [H3]. Thus, the results of this paper will be useful in extending the results in [H3] to more general static spacetimes.

Section 1 shows how causality properties of M and V are related. Section 2 explores various G -invariant constructions in V which yield information about the causal boundary of M ; much of that will be discussed in the more general context

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of chronological sets. Section 3 explores the simplified relations that obtain when M and V are restricted to have spacelike boundaries.

1: SPACETIME COVERING PROJECTIONS AND CAUSALITY CONDITIONS

Let V be a spacetime, that is to say, a connected Lorentz manifold with a time-orientation; let G be a group of time-orientation-preserving isometries of V acting freely and properly discontinuously on V , that is to say, for every point $p \in V$ there is a neighborhood U of p such that for every $g \in G$ except the identity element e , $(g \cdot U) \cap U = \emptyset$. Then $M = V/G$ is also a connected manifold and inherits a Lorentz metric and time-orientation from V , making M a spacetime also. The chronology relation in either spacetime will be denoted by \ll , the causality relation by \prec . Let $\pi : V \rightarrow M$ be the natural projection, carrying a point p to its equivalence class, *i.e.*, its orbit under G , $G \cdot p$. Then π is a principle (or regular) covering projection *i.e.*, the fibre is a group yielding the deck transformations of the covering projection; and that is the way in which any principle covering projection between spacetimes can be constructed.

As indicated before, we could just have easily begun with a spacetime M and let $V = \widetilde{M}/H$, where \widetilde{M} is the universal covering space of M and H is any normal subgroup of $\pi_1(M)$; then $G = \pi_1(M)/H$ is the fibre-group of the projection.

Somewhat more generally, we could start with a spacetime M and let $V = \widetilde{M}/H$ for H any subgroup of $G = \pi_1(M)$, and let $\pi : V \rightarrow M$ be defined by $\pi : H \cdot z \mapsto G \cdot z$ for $z \in \widetilde{M}$ (identifying points in M or V with, respectively, the G - or H -orbits of points in \widetilde{M}). This is a covering projection (not principle unless H is normal in G), and any covering projection between spacetimes arises in this manner. The fibre of the projection is G/H . We have an equivalence relation \cong defined on V by $p \cong q$ if and only if p and q are in the same fibre (*i.e.*, map to the same point by π); we may call this the fibre equivalence relation.

(It is somewhat more awkward to recast this situation as starting with V : One must start with an equivalence relation \cong on V with every point p having a neighborhood U with no two points of U being related and with $\{r \in V \mid r \cong q \text{ for some } q \in U\}$ consisting of a collection of disjoint open sets each isometric to U . The more usual approach in topology is to start with the notion of a covering projection $\pi : V \rightarrow M$.)

The central idea in all these formulations of a covering projection $\pi : V \rightarrow M$ between spacetimes is that π locally be a time-orientation-preserving isometry. Let us call such a map a *spacetime covering projection*; V is the total space, M the base space. The most fundamental (and obvious) relation obtaining in such a context is this:

Proposition 1.1. *Let $\pi : V \rightarrow M$ be a spacetime covering projection. For p and q in V , $p \ll q$ implies $\pi(p) \ll \pi(q)$; for x and y in M , $x \ll y$ implies for any $p \in \pi^{-1}(x)$, there is some $q \in \pi^{-1}(y)$ with $p \ll q$. Similar results hold for \prec .*

Proof. If \bar{c} is a future-directed timelike curve in V , then $\pi \circ \bar{c}$ is a future-directed timelike curve in M . Thus, a future-timelike curve from p to q in V projects to a future-timelike curve from $\pi(p)$ to $\pi(q)$.

Conversely, a future-directed timelike curve c in M has a unique lift \bar{c} to a curve in V starting from any given point in the pre-image of the beginning-endpoint of c , and \bar{c} is a future-directed timelike curve. Thus, a future-timelike curve from x to

y in M lifts to a future-timelike curve from any p sitting above x to some q sitting above y .

Similarly for causal curves and the causality relation. \square

For a principle covering projection with fibre group G , we can recast this: For p and q in V , $\pi(p) \ll \pi(q)$ (respectively, $\pi(p) \prec \pi(q)$) if and only if for some $g \in G$, $p \ll g \cdot q$ (respectively, $\pi(p) \prec g \cdot \pi(q)$).

The idea of this section is to explore, in respect of various causality conditions, the relation between the condition obtaining on the total space and the base space of a spacetime covering projection. As a general rule: A causality condition on the base space holds if and only if a somewhat stronger property holds on the total space.

Surely the first conditions to be examined are those of chronology and causality.

Proposition 1.2. *Let $\pi : V \rightarrow M$ be a spacetime covering projection between spacetimes with fibre equivalence relation \cong . Then M is chronological (respectively, causal) if and only if for all $p, q \in V$, if $p \cong q$, then $p \not\ll q$ (respectively, $p \not\prec q$).*

Proof. If $p \cong q$ in V , then $\pi(p) = \pi(q)$; call this point (in M) x . Then, by Proposition 1.1, if $p \ll q$, we have $x \ll x$, so M is not chronological; and if $p \prec q$, we have $x \prec x$, so M is not causal.

Conversely, suppose for some $x \in M$, $x \ll x$, i.e., there is a timelike curve c from x to x ; say $c : [0, 1] \rightarrow M$, future-directed, with both endpoints at x . Pick a point $p \in \pi^{-1}(x)$. Then c lifts to a future-directed timelike curve $\bar{c} : [0, 1] \rightarrow V$ with $\bar{c}(0) = p$ and $\pi(\bar{c}(t)) = c(t)$ for all t . In particular, $q = \bar{c}(1)$ lies in $\pi^{-1}(x)$, so $q \cong p$; then $p \ll q$. Similarly, if $x \prec x$, then $p \prec q$. \square

It may be worth noting that in the case of a principle covering projection, with the fibre equivalence relation given by the action of a fibre group G , the condition on V in Proposition 1.2 amounts to saying that no point p is chronologically (or causally) related to any member of its orbit $G \cdot p$.

Strong causality may be the next most widely applicable causality condition.

Proposition 1.3. *Let $\pi : V \rightarrow M$ be a spacetime covering projection. Then M is strongly causal if and only if for all $p \in V$, there is a fundamental neighborhood system $\{U_n\}$ for p such that for each n , no causal curve can have one endpoint in U_n and another endpoint in a component U'_n of $\pi^{-1}(\pi[U_n])$ unless $U'_n = U_n$ and the curve remains wholly within U_n .*

An equivalent condition on the fundamental neighborhood system $\{U_n\}$ is that for each n , no causal curve may exit and re-enter U_n , π restricts to a homeomorphism on U_n , and for any two components U'_n and U''_n of $\pi^{-1}(\pi[U_n])$, no point of U'_n is timelike related to any point of U''_n .

Proof. Suppose $p \in V$ has such a fundamental neighborhood system $\{U_n\}$; then $\{W_n = \pi(U_n)\}$ is a fundamental neighborhood system for $x = \pi(p)$, if we restrict n to be so large that each U_n is carried homeomorphically by π to its image W_n . Any causal curve c in M which starts and ends in some W_n lifts to a timelike curve \bar{c} in V , beginning in U_n and ending in some component U'_n of $\pi^{-1}(\pi[U_n])$. Then $U'_n = U_n$ and \bar{c} remains within U_n ; thus, c does not exit W_n . Therefore, $\{W_n\}$ is the required neighborhood system showing strong causality at x .

Conversely, suppose strong causality holds at $x \in M$, i.e., x has a fundamental neighborhood system $\{W_n\}$ with no causal curve in M exiting and re-entering any

W_n . Let p be any point in $\pi^{-1}(x)$. For n sufficiently large, each $\pi^{-1}(W_n)$ has exactly one component U_n containing p with $\pi : U_n \rightarrow W_n$ a homeomorphism; these $\{U_n\}$ form a fundamental neighborhood of p . For any causal curve \bar{c} in V with one endpoint in U_n and another endpoint in any component U'_n of $\pi^{-1}(W_n)$, $c = \pi \circ \bar{c}$ is a causal curve in M with both endpoints in W_n . Then c must remain wholly within W_n ; hence, \bar{c} remains wholly within U_n , whence $U'_n = U_n$.

The condition in the second paragraph is equivalent, since a causal curve exiting one component and entering another implies a timelike relation between points of the components. \square

In the case of a principle covering projection with fibre group G , the condition on V above says that each point has a fundamental neighborhood system $\{U_n\}$ with no causal curve exiting and re-entering any U_n and no points of U_n and $g \cdot U_n$ being timelike related unless $g = e$.

The next result concerns global hyperbolicity. We will use $J(x, y)$ to denote $J^+(x) \cap J^-(y)$, where J^+ and J^- denote causal future and past, respectively (recall that global hyperbolicity is equivalent to strong causality plus each $J(x, y)$ being compact). For this result, very different arguments will be used in the two directions: Arguing from the total space to the base space, the sort of direct methods employed above work for consideration of the spaces $J(x, y)$; but for arguing from the base space to the total space, it is far easier to use the equivalent condition ([HE], section 6.6) that the spacetime has a Cauchy surface.

Proposition 1.4. *Let $\pi : V \rightarrow M$ be a spacetime covering projection. Then M is globally hyperbolic if and only if*

- (1) *V is globally hyperbolic,*
- (2) *every point $p \in V$ has a fundamental neighborhood system as in Proposition 1.3, and*
- (3) *for any $p \in V$, for all $q \gg p$, $J^+(p) \cap \pi^{-1}(\pi(q))$ is finite.*

Proof. Suppose that M is globally hyperbolic. That means it has a Cauchy surface Σ . Using a technique from [GH] (Lemma 4.1), we will see that Σ gives rise to a Cauchy surface $\bar{\Sigma}$ in V , showing that V also is globally hyperbolic (but we must fill in the gap in [GH], which failed to show why $\bar{\Sigma}$ must be connected).

Let $\bar{\Sigma} = \pi^{-1}(\Sigma)$; then $\bar{\Sigma}$ is a topological spacelike hypersurface in V (just as Σ is in M), and every endless causal curve $\bar{c} : (-\infty, \infty) \rightarrow V$ intersects $\bar{\Sigma}$ exactly once: The projection $c = \pi \circ \bar{c} : (-\infty, \infty) \rightarrow M$ is an endless causal curve so it intersects Σ exactly once. Since for any t , $c(t) \in \Sigma$ if and only if $\bar{c}(t) \in \bar{\Sigma}$, this means there is exactly one t_0 with $\bar{c}(t_0) \in \bar{\Sigma}$. Thus, all we need to conclude that $\bar{\Sigma}$ is a Cauchy surface is that it be connected.

Consider any point $x \in \Sigma$ and two points p and q in $\pi^{-1}(x)$. Let $\bar{\sigma}$ be a curve in V from p to q (we are assuming that V is connected); then $\sigma = \pi \circ \bar{\sigma}$ is a closed curve in M containing x . There is a homotopy h , fixing x , from σ to a closed curve σ' lying wholly in Σ (this is because, topologically, $M = \mathbb{R} \times \Sigma$ —see [HE], Proposition 6.6.8; if, say, $\sigma(t) = (s(t), y(t))$ with $x = (0, y(0))$, then let $h(t, u) = (us(t), y(t))$, $0 \leq u \leq 1$). Then, by the homotopy lifting property of covering projections, there is a lift of h to a homotopy \bar{h} in V , and \bar{h} is a homotopy from $\bar{\sigma}$ to some curve $\bar{\sigma}'$, with the same endpoints as σ , lying wholly in $\bar{\Sigma}$. Therefore, p and q are in the same component of $\bar{\Sigma}$.

This shows that all the elements of $\bar{\Sigma}$ sitting over the same point in Σ are in the same component of $\bar{\Sigma}$. Let C be any such component; then $\pi(C)$ is open in Σ (since $\pi : \bar{\Sigma} \rightarrow \Sigma$ is a local homeomorphism) and connected, hence, a component of Σ . Since Σ is connected, $\pi(C)$ must be all of Σ , so C includes points sitting above each point of Σ . That means C is all of $\bar{\Sigma}$, *i.e.*, $\bar{\Sigma}$ is connected. This shows V is globally hyperbolic.

Since M is strongly causal (being globally hyperbolic), we know from Proposition 1.3 that V must satisfy condition (2).

For condition (3), suppose there is a point $p \in V$ above a point $x \in M$ and that there is an infinite number of points $\{q_n\}$ in V , all above the same point $y \in M$, with all $q_n \succ p$. Then we have future-causal curves \bar{c}_n from p to q_n ; each yields a future-causal curve $c_n = \pi \circ \bar{c}_n$ from x to y . If any two of these curves in M , c_n and c_m , were homotopic, then their lifts \bar{c}_n and \bar{c}_m in V would have the same endpoints; but they don't, since q_n and q_m are distinct for $n \neq m$. Thus, the curves $\{c_n\}$ represent distinct homotopy classes in M .

However, M being globally hyperbolic, the causal curves from x to y form a compact set in the compact-open topology ([HE], Proposition 6.6.2); thus, there is a future-causal curve c from x to y and a subsequence $\{c_{n_k}\}$ approaching c in this topology. In particular, for any neighborhood U in M of the image of c , an infinite number of the $\{c_n\}$ lie in U . Let us parametrize c on $[0, 1]$; then we can choose U to be a small tubular neighborhood of the form $(-\epsilon, 1 + \epsilon) \times B^{n-1}$ (where n is the dimension of M and B denotes the Euclidean ball about the origin $\mathbf{0}$), with x corresponding to $(0, \mathbf{0})$ and y to $(1, \mathbf{0})$. Then any causal curve c' in U from x to y must be homotopic to c (with U sufficiently small, c' must lie in $[0, 1] \times B^{n-1}$, hence, be parametrizable as $(t, z(t))$ for $z(t) \in B^{n-1}$). Thus, for any n and m with c_n and c_m both lying in U , both are homotopic to c , hence, to one another—a contradiction. Therefore, condition (3) is true.

Conversely, suppose conditions (1), (2), and (3) are true. By Proposition 1.3, we know that M is strongly causal; we just need to show that for every x and y in M , $J(x, y)$ is compact. Let p be any lift of x to V . For any point $z \in J(x, y)$, there is a future-causal curve c from x to y passing through z ; we can lift that to a future-causal curve \bar{c} in V starting at p and including a point r sitting above z . Then \bar{c} terminates at some point q sitting above y , and we have $r \in J(p, q)$. Therefore, every point in $J(x, y)$ can be expressed as the projection of a point in $J(p, q)$ for some $q \in \pi^{-1}(y)$. But there are only finitely many points $\{q_1, \dots, q_k\}$ sitting above y for which each $J(p, q_i)$ is non-empty. Thus, $J(x, y) = \pi \left(\bigcup_{i \leq k} J(p, q_i) \right)$; since each $J(p, q_i)$ is compact (V being globally hyperbolic) and the continuous image of a compact set is compact, this means $J(x, y)$ is compact. \square

Clearly, condition (3) can be replaced by a similar statement involving J^- . In the case of a principle covering projection with fibre group G , condition (3) says that for any p and q in V , $J^+(p) \cap (G \cdot q)$ is finite.

It's not entirely clear that condition (3) is needed in Proposition 1.4, *i.e.*, although it is implied by M being globally hyperbolic, it may be that it is a consequence of conditions (1) and (2) anyway. Since we can always formulate the context in terms of group actions on the universal covering space, the question comes down to this: If V is a globally hyperbolic spacetime acted on freely and properly discontinuously by a group G of time-orientation-preserving isometries, is it possible that there is a point p and a point q with $G \cdot q$ having an infinite number of elements in

$J^+(p)$?

2: DISCRETE GROUP ACTIONS AND THE FUTURE CAUSAL BOUNDARY

The discussion in this section will focus on the chief aspects of the causal boundary of [GKP]—specifically, the future causal (pre-)boundary as formulated first in [H1] (for categorical construction) and then in [H2] (for topology). As shown in [H1], the full causal boundary has innate difficulties that make it resistant to categorical formulation. But by restricting to the future causal boundary we can formulate future-completion in a manner that applies to spacetimes as merely one example of a much broader category of “chronological sets”; then the process of future-completion (adding the future causal boundary) within this larger category is fully categorical (*i.e.*, functorial) and also universal (hence, unique) in the sense of category theory.

The topology of the causal boundary as formulated in [GKP] cannot be implemented with just the future causal boundary (and it has severe problems of its own). But a topology for any chronological set was developed in [H2], and this topology was shown also to have categorical and universal properties, at least in the case of spacelike boundaries.

The intent of this section is to explore the future causal boundary for spacetimes with discrete group actions. However, the constructions naturally lend themselves to exposition within the broader category of chronological sets. As it turns out, this is a good thing, for the slickest presentation of the results lies in a construction with chronological sets that takes us far afield from spacetimes, even when that is what we start with. A return to constructions more narrowly focused in spacetimes will be the subject of section 3. Subsection 2.1 will look purely at basic properties of chronological sets, while Subsection 2.2 will consider aspects of topology derived from the chronology relation.

2.1: Discussion of the Chronological Relation.

We will make use of some definitions and results from [H1], recalled here:

A *chronological set* is a set X together with a relation \ll (the *chronology relation*) obeying the following:

- (1) \ll is transitive: $x \ll y$ and $y \ll z$ implies $x \ll z$
- (2) \ll is non-reflexive: for all x , $x \not\ll x$
- (3) There are no isolates: for all x , the past of x , $I^-(x) = \{y \mid y \ll x\}$, or the future of x , $I^+(x) = \{y \mid x \ll y\}$, (or both) is non-empty.
- (4) X is separable: There is a countable subset $S_0 \subset X$ such that for $x \ll y$, there is some $s \in S_0$ with $x \ll s \ll y$. (We can say S_0 is “chronologically dense” in X .)

A *past set* in X is a subset $P \subset X$ such that $P = I^-[P]$ (where $I^-[A] = \bigcup_{a \in A} I^-(a)$); any set of the form $I^-[A]$ is a past set. A past set is *indecomposable* if it is not the union of two past sets which are proper subsets; a past set P is indecomposable if and only if for any x and y in P , there is some $z \in P$ with $x \ll z$ and $y \ll z$. A *future chain* is a sequence $\{x_n \mid n \geq 1\}$ with $x_n \ll x_{n+1}$, all n (this plays the role of a timelike curve in a spacetime). For any future chain c , $I^-[c]$ is an IP (indecomposable past set), and for every IP P , there is a future chain c in P such that $P = I^-[c]$. A point $x \in X$ is a *future limit* of the future chain c if $I^-(x) = I^-[c]$ (in a spacetime, this is the same as the topological limit of the sequence); a future limit

of a future chain is unique if X is *past-distinguishing* ($I^-(x) = I^-(y)$ implies $x = y$). X is *future-complete* if every future chain has a future limit. A map $f : X \rightarrow Y$ between chronological sets is *future-continuous* if $x_1 \ll x_2$ implies $f(x_1) \ll f(x_2)$ and if x being a future limit of the future chain $\{x_n\}$ implies $f(x)$ is a future limit of $\{f(x_n)\}$. Let $\hat{\partial}(X) = \{P \subset X \mid P \text{ is an IP and } P \text{ is not } I^-(x) \text{ for any } x \in X\}$ (the *future chronological boundary* of X) and let $\hat{X} = X \cup \hat{\partial}(X)$ (the *future completion* of X), and extend \ll to \hat{X} via (for $x \in X$, $P \in \hat{\partial}(X)$, $Q \in \hat{\partial}(X)$)

- (1) $x \ll P$ iff $x \in P$
- (2) $P \ll x$ iff $P \subset I^-(w)$ for some $w \ll x$
- (3) $P \ll Q$ iff $P \subset I^-(w)$ for some $w \in Q$.

Then \hat{X} is a future-complete chronological set (past-distinguishing if X is), and the inclusion $\iota_X : X \rightarrow \hat{X}$ is future-continuous.

X is *past-determined* if $x \ll y$ whenever $I^-(x)$ is non-empty and $I^-(x) \subset I^-(y)$ for some $w \ll y$ (example: globally hyperbolic spacetimes are past-determined); future-completion preserves being past-determined. For any future-continuous map $f : X \rightarrow Y$ with Y past-determined and past-distinguishing, there is a unique future-continuous map $\hat{f} : \hat{X} \rightarrow \hat{Y}$ extending f (i.e., $\hat{f} \circ \iota_X = \iota_Y \circ f$). This amounts to a functor $\hat{}$ on the category of past-determined, past-distinguishing chronological sets (morphisms being the future-continuous maps), with target the sub-category of future-complete sets; it obeys the universality property that for any $f : X \rightarrow Y$ in the category with Y future-complete, there is a unique extension of f to \hat{X} . For chronological sets which are not past-determined, we can apply the *past-determination* functor, which adds some \ll relations: X^p is the set X with $x \ll y$ defined to hold if and only if, within X , $x \ll y$ or, with $I^-(x)$ non-empty, $I^-(x) \subset I^-(y)$ for some $w \ll y$. Then X^p is a past-determined chronological set (past-distinguishing and future-complete if X is). The GKP future causal boundary of a strongly causal spacetime X , added to X , results in precisely $(\hat{X})^p$, and this is canonically isomorphic to $\widehat{X^p}$. (That canonical isomorphism—a natural equivalence in the categorical sense—is the only really difficult part of the whole enterprise; see [H1], Proposition 13.)

What we are interested in here is a chronological set X with an action from a group G . We need the action to consist of isomorphisms of the chronology relation, i.e., $g \cdot x \ll g \cdot y$ iff $x \ll y$ (for the covering maps of the previous section, we also needed the action to be free—for any $g \in G$ with $g \neq e$, for all $x \in X$, $g \cdot x \neq x$ —but that is not required for the matters in this section). We need to know, first of all, when it is that X/G , the set of G -orbits, inherits the structure of a chronological set; for $G \cdot x$ representing the G -orbit of a point x , we want $G \cdot x \ll G \cdot y$ if and only if $x \ll g \cdot y$ for some $g \in G$. (We will sometimes need to think of $G \cdot x$ as a single element of X/G and sometimes need to think of it as the several elements in the orbit of x ; the context should make it clear.) Then we want to find a way to describe $\widehat{X/G}$ and $\hat{\partial}(X/G)$ in terms of constructions on X .

The first matter is easy:

Proposition 2.1. *Let X be a chronological set with a group action via chronological isomorphisms from a group G . Then the set X/G inherits the structure of a chronological set if and only if for all $g \in G$ and $x \in X$, $x \not\ll g \cdot x$. When this obtains, the projection $\pi : X \rightarrow X/G$ is future-continuous.*

Proof. X/G is necessarily transitive ($x \ll g \cdot y$ and $y \ll h \cdot z$ yield $g \cdot y \ll g \cdot (h \cdot y) = (gh) \cdot z$, so $x \ll (gh) \cdot z$). Since $x \ll y$ implies $G \cdot x \ll G \cdot y$, if x has a non-empty past (or future), so does $G \cdot x$. If S_0 is the countable set making X separable, then $\pi[S_0] = \{G \cdot s \mid s \in S_0\}$ is a countable collection of elements of X/G and serves the same purpose in that set. The only thing that could go wrong is non-reflexivity: $G \cdot x \ll G \cdot x$ occurs if and only if $x \ll g \cdot x$ occurs.

Assume now that $x \ll g \cdot x$ never occurs. We already know that $x \ll y$ implies $\pi(x) \ll \pi(y)$. Suppose $c = \{x_n\}$ is a future chain in X and $I^-(x) = I^-[c]$; we want to show that $I^-(\pi(x)) = I^-[\pi(c)]$. We have $\pi(y) \ll \pi(x)$ means $y \ll g \cdot x$ for some g , whence $g^{-1} \cdot y \ll x$, so $g^{-1} \cdot y \ll x_n$ for some n , so $y \ll g \cdot x_n$, i.e., $\pi(y) \in I^-[\pi(c)]$. Conversely, $\pi(y) \in I^-[\pi(c)]$ means $y \ll g \cdot x_n$ for some n and some g , so $g^{-1} \cdot y \ll x_n$, so $g^{-1} \cdot y \ll x$, so $y \ll g \cdot x$ and $\pi(y) \ll \pi(x)$. \square

For X a chronological set, we will say that a group G has a *chronological action* on X if it acts on X through chronological isomorphisms and in accord with the property in Proposition 2.1. Then we can express the chronology relation on X/G in terms of X and the projection $\pi : X \rightarrow X/G$ thus: For p and q in X/G , $p \ll q$ if and only if some (hence, every) element of $\pi^{-1}(p)$ is in the past of some element of $\pi^{-1}(q)$, which happens if and only if some (hence, every) element of $\pi^{-1}(q)$ lies in the future of some element of $\pi^{-1}(p)$. For a subset $A \subset X$, we then have $\pi[I^-[A]] = I^-[\pi[A]]$. (The convention with brackets vs. parentheses employed here is that for a function $f : X \rightarrow Y$, brackets are used to denote the application of f to a subset of X , i.e., for $A \subset X$, $f[A] = \{f(x) \mid x \in A\}$.) The reason is thus: If $x \ll a \in A$, then $\pi(x) \ll \pi(a)$; and if $p \ll \pi(a)$ for $a \in A$, then for some $x \in \pi^{-1}(p)$, $x \ll a$. Therefore, we can characterize the chronological set X/G by $I_{X/G}^-(\pi(x)) = \pi[I_X^-(x)]$, since $I_{X/G}^-(\pi(x)) = \pi[I_X^-[G \cdot x]] = \pi[\bigcup G \cdot I_X^-(x)]$ (for $S \subset X$, $G \cdot S = \{g \cdot s \mid g \in G\}$, will denote the set of G -orbits of S , so $\bigcup G \cdot S$ denotes the set of points of X in all those G -orbits).

Let X be a chronological set with a chronological action from a group G . To investigate the IPs and the future chronological boundary in X/G , we need to look at something a bit different in X : We will call a subset $A \subset X$ *G-invariant* if $\bigcup G \cdot A = A$. Note that $\bigcup G \cdot I^-[A] = I^-[\bigcup G \cdot A]$ (for $a \in A$ and $x \ll a$, we have $g \cdot x \ll g \cdot a$, so $g \cdot x \in I^-[\bigcup G \cdot A]$; for $x \ll g \cdot a$, we have $x = g \cdot (g^{-1} \cdot x)$ and $g^{-1} \cdot x \ll a$, so $x \in \bigcup G \cdot I^-[A]$); hence, a G -invariant past set is anything of either of these forms. A *G-indecomposable past set* (or a GIP, for “group-indecomposable past set”, even if another letter is used to name the group) will be a G -invariant past set which is not the union of two G -invariant past sets which are proper subsets. Finally, let the *future chronological G-boundary* of X , denoted $\hat{\partial}_G(X)$, consist of $\{ \text{GIPs } A \mid \text{for all } x \in X, A \neq G \cdot I^-(x) \}$.

GIPs will be the basic building blocks for our analysis of G -invariant structures in X as they reflect the items of interest in X/G , as established by the following lemma. (It will develop that $\hat{\partial}_G(X)$ models $\hat{\partial}(X/G)$.)

Lemma 2.2. *Let X be a chronological set with a chronological action from a group G with $\pi : X \rightarrow X/G$ the natural projection. For every GIP A in X , $\pi[A]$ is an IP in X/G , and for every IP P in X/G , $\pi^{-1}[P]$ is a GIP in X .*

Furthermore, an IP P in X/G is an element of the future chronological boundary $\hat{\partial}(X/G)$ if and only if $\pi^{-1}[P]$ is an element of the future chronological G -boundary $\hat{\partial}_G(X)$.

Proof. Since for $x \in X$, $I^-(\pi(x)) = \pi[I^-(x)]$, we have for any $A \subset X$, $I^-[\pi[A]] = \pi[I^-[A]]$. Similarly, for any $p \in X/G$, we have $I^-[\pi^{-1}(p)] = \pi^{-1}[I^-(p)]$, so for any $P \subset X/G$, we have $I^-[\pi^{-1}[P]] = \pi^{-1}[I^-[P]]$.

Now let A be a past set in X ; then $I^-[\pi[A]] = \pi[I^-[A]] = \pi[A]$, showing $\pi[A]$ is a past set. And if P is a past set in X/G , then $I^-[\pi^{-1}[P]] = \pi^{-1}[I^-[P]] = \pi^{-1}[P]$, showing $\pi^{-1}[P]$ is a past set; furthermore, as is evident, $\pi^{-1}[P]$ is G -invariant.

Now let A be a past set in X with $\pi[A]$ decomposable into proper past sets, $\pi[A] = P_1 \cup P_2$; then $A = \pi^{-1}[P_1] \cup \pi^{-1}[P_2]$, and each $\pi^{-1}[P_i]$ is a G -invariant past subset, as well as a proper subset of A . In particular, if A is a GIP in X , then $\pi[A]$ is an IP in X .

Let P be a past set in X/G with $\pi^{-1}[P]$ decomposable into proper G -invariant past sets $\pi^{-1}[P] = A_1 \cup A_2$; then $P = \pi[A_1] \cup \pi[A_2]$, and each $\pi[A_i]$ is a past set, as well as a proper subset of P . Thus, if P is an IP in X/G , then $\pi^{-1}[P]$ is a GIP in X .

An IP P in X/G is an element of $\hat{\partial}(X/G)$ if and only if for no $p \in X/G$ is $P = I^-(p)$. We have $P = I^-(p)$ implies $\pi^{-1}[P] = \pi^{-1}[I^-(p)] = I^-[\pi^{-1}(p)]$; picking any $x \in \pi^{-1}(p)$, this yields $\pi^{-1}[P] = I^-[G \cdot x] = G \cdot I^-(x)$. On the other hand, if $\pi^{-1}[P] = G \cdot I^-(x)$, then $P = \pi[G \cdot I^-(x)] = \pi[I^-(x)] = I^-(\pi(x))$. \square

Thus we know that GIPs are the items of interest in X for analyzing X/G . But what are these new creatures, and how do they work? It turns out they are not new at all, but just the IPs of a new chronological set, X_G , with $\hat{\partial}_G(X) = \hat{\partial}(X_G)$.

For any chronological set X with a free chronological action from a group G , let us define a new relation \ll_G on X , the G -expansion of the chronological relation \ll on X , via $x \ll_G y$ if and only if for some $g \in G$, $x \ll g \cdot y$, i.e., if and only if $\pi(x) \ll \pi(y)$ in X/G . Let X_G denote the partially ordered set (X, \ll_G) , called the G -expansion of (X, \ll) (when no confusion results, we will let X stand for (X, \ll) , as before).

Proposition 2.3. *Let X be a chronological set with a chronological action from a group G .*

- (a) *The G -expansion X_G is a chronological set.*
- (b) *The natural injection $i_G : X \rightarrow X_G$ and natural projection $\pi_G : X_G \rightarrow X/G$ (respectively the same as the identity map and π on the set-level) are future-continuous.*
- (c) *The GIPs of X are precisely the IPs of X_G .*
- (d) *The GIPs of X which are elements of $\hat{\partial}_G(X)$ are precisely the IPs of X_G which are elements of $\hat{\partial}(X_G)$, i.e., $\hat{\partial}_G(X) = \hat{\partial}(X_G)$.*

Proof. All the properties of a chronological set follow directly from Proposition 2.1, as does the future-continuity of π_G .

Let I_G^- denote the past in X_G . We know that for $x \in X$, $I_G^-(x) = G \cdot I^-(x)$; then for $A \subset X$, $I_G^-[A] = G \cdot I^-[A]$.

For the future-continuity of i_G , consider x a future limit of a future chain c in X , i.e., $I^-(x) = I^-[c]$. Then $I_G^-(x) = G \cdot I^-(x) = G \cdot I^-[c] = I_G^-[c]$.

The G -invariant past sets in X are precisely the sets of the form $G \cdot I^-[A]$; since $G \cdot I^-[A] = I_G^-[A]$, these are precisely the past sets of X_G . Thus, indecomposability in the one form amounts to indecomposability in the other form, yielding the GIPs of X the same as the IPs of X_G . A GIP which has the form $G \cdot I^-(x)$ is precisely one which has the form $I_G^-(x)$. \square

The G -expansion X_G must be treated with care, as it is a rather anomalous chronological set even if X is very well-behaved (such as being a strongly causal spacetime): Past-distinguishing fails widely in X_G , as for every $x \in X$, for all $g \in G$, $I_G^-(g \cdot x) = I_G^-(x)$.

Just as IPs in spacetimes can be characterized as the pasts of timelike curves, and IPs in chronological sets can be characterized as the pasts of future chains, GIPs in chronological sets with a chronological G -action can be characterized as the pasts of what might be thought of as “ G -invariant future chains” (in spacetimes: “ G -invariant timelike curves”, in analogous formation). What this really means is the G -orbit of a future chain; this will be written as $G \cdot c$ for convenience, though it may be argued that $\bigcup G \cdot c$ is more proper.

Proposition 2.4. *Let X be a chronological set with a chronological action from a group G . For any future chain c in X , $I^-[G \cdot c]$ is a GIP, and every GIP can be written in that form.*

Proof. Let c be a future chain in X ; then $I^-[G \cdot c] = \bigcup G \cdot I^-[c] = I_G^-[c]$. Since c is also a future chain in X_G , $I_G^-[c]$ is an IP in X_G , hence, a GIP in X .

Any GIP A , being an IP in X_G , can be written as $I_G^-[c]$ for some future chain $c = \{x_n\}$ in X_G . Let $x'_1 = x_1$. Suppose we’ve defined $x'_1 \ll \dots \ll x'_n$ with $x'_i = g_i \cdot x_i$ for some $g_i \in G$. We have $x_n \ll_G x_{n+1}$, so $x_n \ll g \cdot x_{n+1}$ for some $g \in G$; then $x'_n = g_n \cdot x_n \ll g_n g \cdot x_{n+1}$, and let $x'_{n+1} = g_n g \cdot x_{n+1}$. This defines $c' = \{x'_n\}$, a future chain in X , with $G \cdot c' = G \cdot c$. Then $A = I_G^-[c] = \bigcup G \cdot I^-[c] = I^-[G \cdot c] = I^-[G \cdot c']$. \square

Corollary 2.5. *For any IP P in X , $\bigcup G \cdot P$ is a GIP, and every GIP can be so expressed.*

Proof. Any IP can be written as $I^-[c]$ for some future chain c ; then $\bigcup G \cdot I^-[c] = I^-[G \cdot c]$, which is a GIP by Proposition 2.4. Conversely, by Proposition 2.4, any GIP can be written as $I^-[G \cdot c]$ for some future chain c , and $I^-[G \cdot c] = \bigcup G \cdot I^-[c]$; $I^-[c]$ is an IP. \square

Following on from Proposition 2.3(b), we can use π_G to relate $\hat{\partial}_G(X)$, viewed as $\hat{\partial}(X_G)$, to $\hat{\partial}(X/G)$ in a direct fashion: From $\pi_G : X_G \rightarrow X/G$ we can construct the extension $\widehat{\pi}_G : \widehat{X}_G \rightarrow \widehat{X/G}$ (recall that \hat{Y} denotes $Y \cup \hat{\partial}(Y)$, the future completion of Y). It was stated in the introductory portion of this section that a future-continuous map $f : Z \rightarrow Y$ yields a future-continuous future-completion $\hat{f} : \hat{Z} \rightarrow \hat{Y}$ when Y is past-determined and past-distinguishing—and X/G might not be past-distinguishing, even if X is. But Proposition 6 of [H1] actually allows for looser conditions.

Proposition 2.6. *Let X be a chronological set with a chronological action from a group G . The natural projection from the G -expansion $\pi_G : X_G \rightarrow X/G$ has a future-continuous extension to the future-completions, $\widehat{\pi}_G : \widehat{X}_G \rightarrow \widehat{X/G}$. It has these further properties:*

- (1) *For any α and β in \widehat{X}_G , $\widehat{\pi}_G(\alpha) \ll \widehat{\pi}_G(\beta)$ iff $\alpha \ll_G \beta$.*
- (2) *$\widehat{\pi}_G$ restricts on the boundaries to an isomorphism of chronology relations and of the subset relation, $\widehat{\pi}_G : \hat{\partial}(X_G) \cong \hat{\partial}(X/G)$.*

Proof. Proposition 6 of [H1] states that in order for a future-continuous map $f : Z \rightarrow Y$ to have a future-continuous extension $\hat{f} : \hat{Z} \rightarrow \hat{Y}$, one doesn’t really need

that Y be past-distinguishing, but only that for any future chain c in Z which generates an element of $\hat{\partial}(Z)$ (i.e., c has no future limit in Z), there not be two distinct future limits of $f[c]$ in Y . The map \hat{f} is then defined thus: For $z \in Z$, $\hat{f}(z) = f(z)$; for c generating an element P of $\hat{\partial}(Z)$, if $f[c]$ has a (single) future limit $y \in Y$, then $\hat{f}(P) = y$; and for c generating $P \in \hat{\partial}(Z)$ with $f[c]$ having no future limit in Y , $\hat{f}(P) = I^-[f[c]]$, an element of $\hat{\partial}(Y)$.

So consider a future chain c in X_G . Suppose $\pi_G[c]$ has a future limit $p = \pi(x)$ in X/G ; then $I^-(\pi(x)) = I^-[\pi[c]]$. But $I^-(\pi(x)) = \pi[I^-(x)]$ and $I^-[\pi[c]] = \pi[I^-[c]]$ (as mentioned in the proof of Lemma 2.2), so we have $\pi[I^-(x)] = \pi[I^-[c]]$; this implies $\bigcup G \cdot I^-(x) = \bigcup G \cdot I^-[c]$. But the first of these is $I_G^-(x)$, the second $I_G^-[c]$; thus, x is a future limit of c in X_G . Therefore, no future chain generating a future boundary element of X_G can have even one future limit for its image in X/G , much less two.

This allows us to define a map $\widehat{\pi}_G : \widehat{X_G} \rightarrow \widehat{X/G}$; but Proposition 6 of [H1] uses past-determination of the codomain to show this map preserves the chronology relation, and we aren't insisting that X/G be past-determined. So we must consider this issue separately:

For $f : Z \rightarrow Y$, past-determination of Y is used only in showing that for $P \in \hat{\partial}(Z)$ and $z \in Z$ with $P \ll z$ and $\hat{f}(P) \in Y$, then $\hat{f}(P) \ll f(z)$ (if $\hat{f}(P) \in \hat{\partial}(Y)$, then $\hat{f}(P) \ll f(z)$ follows anyway, irrespective of past-determination). So consider $P \in \hat{\partial}(X_G)$; the import of what was just shown was that $\widehat{\pi}_G(P)$ lies in $\hat{\partial}(X/G)$ and not in X/G (i.e., if the future chain c generates P in X_G , then $\pi[c]$ has no future limit in X/G).

Preserving the chronology relation was the only issue; past-determination and past-distinguishing play no further role in showing future-continuity.

For property (1), consider x and y in X , $A = \bigcup G \cdot P$ and $B = \bigcup G \cdot Q$ in $\hat{\partial}(X_G)$, where P and Q are in $\hat{\partial}(X)$, generated by future chains c and d in X ; we know that $\widehat{\pi}_G(A)$ is an element of $\hat{\partial}(X/G)$, specifically $I^-[\pi[c]]$. We consider $\alpha \ll_G \beta$ for α and β successively being x and y , x and A , A and y , and A and B :

We have $x \ll_G y \iff \pi(x) \ll \pi(y)$. We have $x \ll_G A \iff x \in A \iff \pi(x) \in \pi[A] = \pi[P] = \pi[I^-[c]] = I^-[\pi[c]]$, i.e., $\pi(x) \ll \widehat{\pi}_G(A)$. We have $A \ll_G y \iff A \subset I_G^-(w)$ for some $w \ll_G y \iff \bigcup G \cdot I^-[c] \subset \bigcup G \cdot I^-(w)$ for some $w \ll_G y \iff \pi[I^-[c]] \subset \pi[I^-(w)]$ for some w with $\pi(w) \ll \pi(y) \iff \widehat{\pi}_G(A) = I^-[\pi[c]] = \pi[I^-[c]] \subset \pi[I^-(w)] = I^-(\pi(w))$ for some $\pi(w) \ll \pi(y) \iff \widehat{\pi}_G(A) \ll \pi(y)$. We have $A \ll_G B \iff A \ll_G x \ll_G B$ for some $x \in X$, which, by the previous results, is equivalent to $\widehat{\pi}_G(A) \ll \widehat{\pi}_G(B)$.

For property (2), in light of property (1), we need only show that $\widehat{\pi}_G$ is a bijection from $\hat{\partial}(X_G)$ to $\hat{\partial}(X/G)$ that preserves subsets in both directions. We already know that $\widehat{\pi}_G$ takes boundary elements to boundary elements. To show it is onto the boundary, consider any $P \in \hat{\partial}(X/G)$ generated by a future chain $\sigma = \{p_n\}$. For each n , pick any $x_n \in \pi^{-1}(p_n)$; then $c = \{x_n\}$ is a future chain in X_G (property (1)). If c has a future limit x in X_G , then (again by property (1)) $\pi(x)$ is a future limit of σ in X/G ; but that's forbidden by P being in $\hat{\partial}(X/G)$. Thus, $A = I_G^-[c] \in \hat{\partial}(X/G)$, and $\widehat{\pi}_G(A) = P$.

Consider how $\widehat{\pi}_G$ works with the subset relation: For elements A and A' of $\hat{\partial}(X_G)$, generated respectively by future chains $\{x_n\}$ and $\{x'_n\}$ in X_G , we have $A \subset A' \iff$ for all n , there is some m with $x_n \ll_G x'_m \iff$ for all n , there

is some m with $\pi(x_n) \ll \pi(x'_m) \iff \widehat{\pi}_G(A) \subset \widehat{\pi}_G(A')$. This also establishes the injectivity of $\widehat{\pi}_G$, restricted to $\hat{\partial}(X_G)$. \square

Proposition 2.6(2) tells us that our object of interest—the future boundary of X/G —is entirely reflected in the future boundary of the somewhat mysterious chronological set X_G . Then Proposition 2.3(d) tells us that that boundary is identifiable with something we can get a handle on—the GIPs of X making up $\hat{\partial}_G(X)$. (Corollary 2.5 shows us how to view GIPs easily in term of IPs.)

But what we really want out of all this is topology.

2.2: Discussion of the chronological topology.

The topology of an intended boundary for a spacetime—or, more generally, chronological set—is addressed in [H2], where what is called the $\hat{\cdot}$ -topology is defined for any chronological set; it may perhaps be called the (future) chronological topology. This differs from the GKP topology on the causal boundary (or even just the future causal boundary). That is intentional, as the GKP topology has some severe problems; notably, the GKP topology for the causal boundary of Minkowski space is not that of the conformal embedding into the Einstein static spacetime, as one would expect. The $\hat{\cdot}$ -topology of [H2] does give the embedding topology for the future causal boundary of Minkowski space, as well as for proper embeddings of any strongly causal spacetime with a spacelike causal boundary (see Theorem 5.3 of [H2]); also, the $\hat{\cdot}$ -topology for the future causal boundary of any standard static spacetime—metric product $\mathbb{L}^1 \times N$ for N a Riemannian manifold—follows directly from a geometric boundary construction on N (see Theorem 6 of [H3]). It also has such nice properties as making a chronological set X dense in $X \cup \partial(X)$ for any sort of reasonable approximation of a future boundary $\partial(X)$ (Theorem 2.4 of [H2]). Accordingly, it is the $\hat{\cdot}$ -topology that will be used for boundaries here.

Note that in the case of a chronological action from a group G on a chronological set X , the group automatically acts via homeomorphisms, using the $\hat{\cdot}$ -topology on X : This is because for any $g \in G$, the motion by g — $\phi_g : X \rightarrow X$ defined by $\phi_g : x \mapsto g \cdot x$ —is an isomorphism of the chronology relation (since $x \ll y$ implies $\phi_g(x) = \phi_g(y)$ and $\phi_{g^{-1}} = (\phi_g)^{-1}$) and the $\hat{\cdot}$ -topology is defined purely in terms of the chronology relation.

The chief virtue of the $\hat{\cdot}$ -topology as used in [H2] is the fact (Proposition 2.7 in [H2]) that in at least one reasonable category, any map $f : Z \rightarrow Y$ between chronological sets which is not only future-continuous but also continuous in the $\hat{\cdot}$ -topologies of Z and Y yields a map $\hat{f} : \hat{Z} \rightarrow \hat{Y}$ between the future-completions which is again not only future-continuous, but continuous in the $\hat{\cdot}$ -topologies. However, the category in which this was proved is that of chronological sets which are past-regular and past-distinguishing and with spacelike future boundaries. Past-regularity is not a problem for our context of $\pi_G : X_G \rightarrow X/G$ (as will be easily shown), and the absence of past-distinguishing we've already dealt with in establishing the existence of $\widehat{\pi}_G$; but the issue of spacelike boundaries is not readily sidestepped (and not desirable to subsume). Our tack must be to look at continuity afresh, without specific reference to Proposition 2.7 of [H2].

Past-regularity is needed in order to use a relatively simple formulation for the $\hat{\cdot}$ -topology (a more complex version being needed without that property being present). Past-regularity has to do with whether IPs are what we naively expect: A point x is called *past-regular* if $I^-(x)$ is an IP, which is equivalent to there being

a future chain c such that x is a future limit of c ; a chronological set is called past-regular if all its points are. Thus, any spacetime is past-regular (just take a future-directed timelike curve c terminating at x ; $I^-(x) = I^-[c]$, an IP), and if X is a past-regular chronological set, so is \hat{X} (for any IP $P \in \hat{\partial}(X)$, just take a future chain c generating P ; $I_{\hat{X}}^-(P) = I_{\hat{X}}^-[c]$, an IP of \hat{X}). Thus, in our context of interest, where X and X/G are both spacetimes, the only object we need be at all concerned about is X_G ; but this is not a worry:

Proposition 2.7. *Let X be a chronological set with a chronological action from a group G . If X is past-regular, then so is X/G , and X_G is past-regular if and only if X/G is.*

Proof. Suppose X is past-regular. For any point $p \in X/G$, pick a point $x \in \pi^{-1}(p)$; then there is a future chain c in X with $I^-[c] = I^-(x)$. Then $\pi[c]$ is a future chain in X/G , and $I^-[\pi[c]] = \pi[I^-[c]] = \pi[I^-(x)] = I^-(\pi(x)) = I^-(p)$, so p is past-regular.

Suppose X_G is past-regular. For any point $p \in X/G$, pick a point $x \in \pi^{-1}(p)$; then there is a future chain c in X_G with $I_G^-[c] = I_G^-(x)$, i.e., $\bigcup G \cdot I^-[c] = \bigcup G \cdot I^-(x)$; from this it follows that $\pi[I^-[c]] = \pi[I^-(x)]$, i.e., $I^-[\pi[c]] = I^-(\pi(x)) = I^-(p)$. Since $\pi[c]$ is a future chain in X/G , this shows p to be past-regular.

Suppose X/G is past-regular. For any point $x \in X$, $\pi(x)$ is the future limit of some chain $\sigma = \{p_n\}$ in X/G , i.e., $I^-[\sigma] = I^-(\pi(x))$. For each n , pick some $x_n \in \pi^{-1}(p_n)$; then $c = \{x_n\}$ is a future chain in X_G . We have $\pi[I^-[c]] = I^-[\pi[c]] = I^-[\sigma] = I^-(\pi(x)) = \pi[I^-(x)]$, from which it follows that $\bigcup G \cdot I^-[c] = \bigcup G \cdot I^-(x)$, i.e., $I_G^-[c] = I_G^-(x)$; thus, x is past-regular in X_G . \square

(It is unclear whether X/G can be past-regular without X also being past-regular; but this does not seem to be of practical concern.)

For a past-regular chronological set X , the $\hat{\cdot}$ -topology is defined thus: Let $\mathcal{IP}(X)$ denote the set of IPs of X . For any sequence of points $\sigma = \{x_n\}$, define $L(\sigma)$ (the “first-order” limits of σ —plural, in case of non-Hausdorff contexts) by $x \in L(\sigma)$ if and only if

- (1) for all $y \ll x$, eventually $y \ll x_n$ (i.e., for n sufficiently large, $y \ll x_n$) and
- (2) for any $P \in \mathcal{IP}(X)$ such that $P \supset I^-(x)$ but $P \neq I^-(x)$, there is some $y \in P$ such that eventually $y \not\ll x_n$.

L is called the limit-operator for X .

We get a topology (the $\hat{\cdot}$ -topology) by defining a set A to be closed if and only if for every sequence σ in A , $L(\sigma) \subset A$. A function $f : X \rightarrow Y$ between chronological sets will be continuous in the $\hat{\cdot}$ -topologies if for every sequence σ in X , $f[L(\sigma)] \subset L(f[\sigma])$ (that is a bit stronger than is needed, in strongly non-Hausdorff cases—a necessary and sufficient condition involves iteration of the set function $L[\cdot]$ an uncountable number of times, up to the first uncountable ordinal—but that is seldom of moment). Details are in [H2].

Proposition 2.8. *Let X be a past-regular chronological set with a chronological action from a group G . The map $\pi_G : X_G \rightarrow X/G$ is continuous in the $\hat{\cdot}$ -topologies, as is its extension $\widehat{\pi}_G : \widehat{X}_G \rightarrow \widehat{X/G}$ to the future completions; furthermore, for every sequence σ and point x in \widehat{X}_G , σ converges to x if $\widehat{\pi}_G[\sigma]$ converges to $\widehat{\pi}_G(x)$.*

Proof. Since X is past-regular, by Proposition 2.7, so are X_G and X/G , so we can use the definition of $\hat{\cdot}$ -topology given by L as above (L will denote the limit-operator in X/G , L_G the one in X_G , with \hat{L} and \hat{L}_G for $\widehat{X/G}$ and \widehat{X}_G respectively).

Note first that Lemma 2.2 and Proposition 2.3(c) combine to yield a correspondence via π between the IPs of X/G and the IPs of X_G (which, by Corollary 2.5, are the G -orbits of IPs of X); specifically, for $A \in \mathcal{IP}(X_G)$, $\pi[A] \in \mathcal{IP}(X/G)$, and for $P \in \mathcal{IP}(X/G)$, $\pi^{-1}[P] \in \mathcal{IP}(X_G)$. Furthermore, this correspondence clearly respects the subset relation: For $A \subset B$ IPs in X_G , $\pi[A] \subset \pi[B]$, and for $P \subset Q$ IPs in X/G , $\pi^{-1}[P] \subset \pi^{-1}[Q]$. Thus, this correspondence is a bijection between $\mathcal{IP}(X_G)$ and $\mathcal{IP}(X/G)$. Proposition 2.6(2) tells us that this correspondence restricts (as $\bar{\pi}_G$) in perfect measure to the future boundaries—and, consequently, non-boundary IPs (*i.e.*, pasts of points) correspond to non-boundary IPs.

Let $\sigma = \{x_n\}$ be a sequence in X and x a point in X . First assume $x \in L_G(\sigma)$; we want to show that $\pi(x) \in L(\pi[\sigma])$, *i.e.*, that it satisfies conditions (1) and (2) of the definition of L . Let $p = \pi(x)$.

- (1) For any $q \ll p$, pick $y \in \pi^{-1}(q)$; then $y \ll_G x$. Consequently, eventually $y \ll_G x_n$, which implies $q \ll \pi(x_n)$.
- (2) For any $P \in \mathcal{IP}(X/G)$ with $P \supsetneq I^-(p)$, we have $A = \pi^{-1}[P] \in \mathcal{IP}(X_G)$ with $A \supsetneq \pi^{-1}[I^-(p)] = I_G^-(x)$. Consequently, there is some $y \in A$ with eventually $y \not\ll_G x_n$, which implies $\pi(y) \not\ll \pi(x_n)$.

This establishes the continuity of $\pi_G : X_G \rightarrow X/G$ (since preservation of the L operator implies continuity).

Now assume $\pi(x) \in L(\pi[\sigma])$; we want to show that $x \in L_G(\sigma)$. (Again, $p = \pi(x)$.)

- (1) For any $y \ll_G x$, we have $\pi(y) \ll p$, so eventually $\pi(y) \ll \pi(x_n)$, which implies $y \ll_G x_n$.
- (2) For any $A \in \mathcal{IP}(X_G)$ with $A \supsetneq I_G^-(x)$, we have $P = \pi[A] \in \mathcal{IP}(X/G)$ with $P \supsetneq I^-(p)$. Consequently, there is some $q \in P$ with eventually $q \not\ll \pi(x_n)$. Pick some $y \in \pi^{-1}(q)$; then $y \in A$ and $y \not\ll_G x_n$.

This establishes the odd sort of “inverse-continuity” of π_G (by the same sort of inductive process which establishes that preservation of L is sufficient for continuity, *i.e.*, preservation of L^Ω).

Now we expand to consideration of the future-completed objects, \widehat{X}_G and $\widehat{X/G}$. The considerations above still suffice for sequences in X and limits in X_G or in X/G , since there is very little additional matter to be considered, and that little works out very easily: For instance, to show $\pi(x) \in \hat{L}(\pi[\sigma])$ (with \hat{L} denoting the limit-operator in $\widehat{X/G}$, we must show additionally that for any $P \in \hat{\partial}(X/G)$ with $P \ll p$, eventually $P \ll \pi(x_n)$; but $P \ll p$ means there is some $q \ll p$ with $P \subset I^-(p)$, and the rest follows. Also, for any past-regular chronological set Y , $\mathcal{IP}(\hat{Y})$ and $\mathcal{IP}(Y)$, while not identical, correspond very nicely: For $P \in \mathcal{IP}(\hat{Y})$, $Y \cap P \in \mathcal{IP}(Y)$, and for $Q \in \mathcal{IP}(Y)$, $\hat{I}^-[Q] \in \mathcal{IP}(\hat{Y})$ (where \hat{I}^- is the past in \hat{Y}), and this is a bijection respecting \subset (this is from the Lemma in Theorem 2.4 of [H2]).

Note that the correspondence of the chronology relations between X_G and X/G , used repeatedly in the arguments above, extends to \widehat{X}_G and $\widehat{X/G}$ as well (see Proposition 2.6(1)). Thus the same arguments apply:

Let $\sigma = \{\alpha_n\}$ be a sequence in \widehat{X}_G and α an element of $\widehat{X/G}$. First assume $\alpha \in \hat{L}_G(\sigma)$; we want to show $\widehat{\pi}_G(\alpha) \in \hat{L}(\widehat{\pi}_G[\sigma])$. Let $\theta = \widehat{\pi}_G(\alpha)$.

- (1) For any $\phi \ll \theta$, there is some $q \in X/G$ with $\phi \ll q \ll \theta$. Pick $y \in \pi^{-1}(q)$;

then $y \ll_G \alpha$. Consequently, eventually $y \ll_G \alpha_n$, which implies $q \ll \widehat{\pi_G}(\alpha_n)$, hence $\phi \ll \widehat{\pi_G}(\alpha_n)$.

- (2) For any $P \in \mathcal{IP}(\widehat{X/G})$ with $P \supsetneq \hat{I}^-(\theta)$, we have $A = \widehat{\pi_G}^{-1}[P] \in \mathcal{IP}(\widehat{X_G})$ with $A \supsetneq \widehat{\pi_G}^{-1}[\hat{I}^-(\theta)] = \hat{I}_G^-(\alpha)$. Consequently, there is some $\beta \in A$ with eventually $\beta \not\ll_G \alpha_n$, which implies $\widehat{\pi_G}(\beta) \not\ll \widehat{\pi_G}(\alpha_n)$.

This establishes the continuity of $\widehat{\pi_G} : \widehat{X_G} \rightarrow \widehat{X/G}$.

Now assume $\widehat{\pi_G}(\alpha) \in \hat{L}(\widehat{\pi_G}[\sigma])$; we want to show $\alpha \in \hat{L}_G(\sigma)$. (Again, $\theta = \widehat{\pi_G}(\alpha)$.)

- (1) For any $\beta \ll_G \alpha$, we have $\widehat{\pi_G}(\beta) \ll \theta$, so eventually $\widehat{\pi_G}(\beta) \ll \widehat{\pi_G}(\alpha_n)$, which implies $\beta \ll_G \alpha_n$.
- (2) For any $A \in \mathcal{IP}(\widehat{X_G})$ with $A \supsetneq \hat{I}_G^-(\alpha)$, we have $P = \widehat{\pi_G}[A] \in \mathcal{IP}(\widehat{X/G})$ with $P \supsetneq \hat{I}^-(\theta)$. Consequently, there is some $\phi \in P$ with eventually $\phi \not\ll \widehat{\pi_G}(\alpha_n)$. There is some $q \in X/G$ with $\phi \ll q \in P$; then $q \not\ll \widehat{\pi_G}(\alpha_n)$. Pick some $y \in \pi^{-1}(q)$; then $y \in A$ and $y \not\ll_G \alpha_n$.

This establishes the inverse-continuity of $\widehat{\pi_G}$. \square

This completes the topological identification of the future chronological boundary of X/G :

Theorem 2.9. *Let X be a past-regular chronological set with a chronological action from a group G . Then $\hat{\partial}(X/G)$, the future chronological boundary of the quotient space, is identifiable with $\hat{\partial}_G(X)$, the future chronological G -boundary of the total space, topologized as $\hat{\partial}(X_G)$, the future chronological boundary of the G -expansion of the total space.*

Specifically: The map $\pi_G^\partial : \hat{\partial}(X_G) \rightarrow \hat{\partial}(X/G)$ is a homeomorphism in the \wedge -topologies, and $\hat{\partial}(X_G)$ is identified with $\hat{\partial}_G(X)$ via any IP in X_G being a GIP in X and vice versa, with boundary elements corresponding to one another. The attachment of $\hat{\partial}(X/G)$ to X/G is exactly reflected in the attachment of $\hat{\partial}_G(X)$ to X —interpreted as $\hat{\partial}(X_G)$ attaching to X_G —via the map $\widehat{\pi_G} : \widehat{X_G} \rightarrow \widehat{X/G}$, in that a sequence σ in $\widehat{X_G}$ converges to an element $A \in \hat{\partial}(X_G)$ if and only if $\widehat{\pi_G}[\sigma]$ converges to $\widehat{\pi_G}(A)$ in X/G .

Proof. The statement about sequences in $\widehat{X_G}$ and $\widehat{\pi_G}$ comes directly from Proposition 2.8. Proposition 2.6 establishes that the restriction of $\widehat{\pi_G}$ to the boundaries is a bijection $\pi_G^\partial : \hat{\partial}(X_G) \rightarrow \hat{\partial}(X/G)$; then the statements about continuity and inverse-continuity of $\widehat{\pi_G}$ amount to saying that π_G^∂ is a homeomorphism. Proposition 2.3 gives the identification of $\hat{\partial}(X_G)$ with $\hat{\partial}_G(X)$. \square

3: QUOTIENTS OF SPACETIMES WITH SPACELIKE BOUNDARIES

How do group actions on a spacetime extend to the boundary, and what is the nature of the interaction between the boundary and the group action? The subject of this section is to explore those questions, especially in the context of spacelike boundaries.

The future chronological boundary of X/G , as a topological space, may be only tenuously related to X , the future chronological boundary of X , and the group action, even in the simplest of settings.

Consider, for instance, Minkowski 2-space \mathbb{L}^2 ; its well-known future boundary $\hat{\partial}(\mathbb{L}^2)$ consists of two null lines (\mathcal{I}_L and \mathcal{I}_R , left and right future null infinity) joined

at a point ($i^+ = \text{future timelike infinity}$); see, e.g., [HE]. There is a simple action from the integers \mathbb{Z} on \mathbb{L}^2 given by $m \cdot (t, x) = (t, x + m)$. The quotient \mathbb{L}^2/\mathbb{Z} is the Minkowski 2-cylinder, the standard static spacetime $\mathbb{L}^1 \times \mathbb{S}^1$ (with a circumference of 1 for the circle \mathbb{S}^1). By Example 2 in [H3], the future chronological boundary of the product spacetime $\mathbb{L}^1 \times K$ is very simple when K is compact: It is just a single point $\{i^+\}$.

The \mathbb{Z} -action on \mathbb{L}^2 extends continuously (though not freely) to the boundary $\mathcal{I}_L \cup i^+ \cup \mathcal{I}_R$. We can readily parametrize \mathcal{I}_L , the IPs of \mathbb{L}^2 generated by null lines rising to the left, as $\{P_a^L \mid a \in \mathbb{R}\}$ with $P_a^L = \{(t, x) \mid t < -x + a\}$; similarly, $\mathcal{I}_R = \{P_a^R \mid a \in \mathbb{R}\}$ with $P_a^R = \{(t, x) \mid t < x + a\}$. Then the homeomorphism $\phi_m : \mathbb{L}^2 \rightarrow \mathbb{L}^2$ defined by $\phi_m : p \mapsto m \cdot p$ extends continuously to $\hat{\partial}(\mathbb{L}^2)$ by $P_a^L \mapsto P_{a+m}^L$, $P_a^R \mapsto P_{a-m}^R$, and $i^+ \mapsto i^+$ (i^+ being the IP consisting of all of \mathbb{L}^2). So we can at least consider the quotient $\hat{\partial}(\mathbb{L}^2)/\mathbb{Z}$; that consists of two circles and a singular point (the equivalence class of i^+) whose only neighborhood in $\hat{\partial}(\mathbb{L}^2)/\mathbb{Z}$ is that entire space: a strongly non-Hausdorff result. This bears no apparent relation to $\hat{\partial}(\mathbb{L}^2/\mathbb{Z})$.

It is worth noting that there is another topology of possible interest on $\hat{\partial}(\mathbb{L}^2)/\mathbb{Z}$, in that $\widehat{\mathbb{L}^2}/\mathbb{Z}$ is a chronological set in its own right, as in Proposition 2.1. This holds generally: As Lemma 3.1 below will show, a chronological action from a group G on a past-regular chronological set X extends to a chronological G -action on \widehat{X} , so that, by Propositions 2.1 and 2.7, \widehat{X}/G is also a past-regular chronological set; it thus has its own $\widehat{}$ -topology. Since the G -action takes X to X , the boundary $\hat{\partial}(X)$ is a G -invariant subset of \widehat{X} , so $\hat{\partial}(X)/G$ is a subspace of \widehat{X}/G and inherits a subspace topology.

In the case of \mathbb{L}^2 , the $\widehat{}$ -topology on $\hat{\partial}(\mathbb{L}^2)/\mathbb{Z}$ (i.e., as a subspace of $(\widehat{\mathbb{L}^2}/\mathbb{Z}, \widehat{}\text{-topology})$), is totally indiscrete: For any two elements P and Q in $\hat{\partial}(\mathbb{L}^2)$, with $\langle P \rangle$ and $\langle Q \rangle$ denoting the equivalence classes under the \mathbb{Z} -action, $\langle P \rangle \in L(\sigma_{\langle Q \rangle})$ with $\sigma_{\langle Q \rangle}$ denoting the constant sequence whose every element is $\langle Q \rangle$. To see this, first note that for any $P \in \hat{\partial}(\mathbb{L}^2)$, $I^-(\langle P \rangle)$ is all of \mathbb{L}^2/\mathbb{Z} , since for any $p \in \mathbb{L}^2$, there is some n such that $p \in n \cdot P$, so $p \ll n \cdot P$, so $\langle p \rangle \ll \langle P \rangle$. It follows that no IP of $\widehat{\mathbb{L}^2}/\mathbb{Z}$ properly contains $I^-(\langle P \rangle)$ (by Lemma 2.2 we know that the IPs of $\widehat{\mathbb{L}^2}/\mathbb{Z}$ are the GIPs of $\widehat{\mathbb{L}^2}$). Thus, to see that $\langle P \rangle \in L(\sigma_{\langle Q \rangle})$, we just need to note that for every $\langle p \rangle \ll \langle P \rangle$, $\langle p \rangle \ll \langle Q \rangle$ also—true, since $I^-(\langle Q \rangle)$ is also \mathbb{L}^2/\mathbb{Z} . In some sense this wholly amorphous topological space is a better reflection of $\hat{\partial}(\mathbb{L}^2/\mathbb{Z})$ than that given by the quotient topology: It accurately suggests that we ought to view all those elements of $\hat{\partial}(\mathbb{L}^2)/\mathbb{Z}$ as really a single point. But it is still not at all the same thing as $\hat{\partial}(\mathbb{L}^2/\mathbb{Z})$.

On the other hand, some spacetimes V with group action G evince a very simple and natural relation between $\hat{\partial}(V/G)$ and the G -action on V extending to $\hat{\partial}(V)$. Consider, for example, “lower” Minkowski space, $\mathbb{L}_-^2 = \{(t, x) \in \mathbb{L}^2 \mid t < 0\}$, with the same \mathbb{Z} -action as above. The IPs of this spacetime are easily calculated, so that $\hat{\partial}(\mathbb{L}_-^2)$ can be seen to be $\{P_a \mid a \in \mathbb{R}\}$ where $P_a = \{(t, x) \mid t < -|x - a|\}$; thus, $\hat{\partial}(\mathbb{L}_-^2) \cong \mathbb{R}$. The GIPs of \mathbb{L}_-^2 are also easily found (just $\bigcup \mathbb{Z} \cdot P$ for any IP P), yielding the boundary-GIPs as $\{B_a \mid a \in \mathbb{R}\}$ with $B_a = \{(t, x) \mid t < -|x - a + m| \text{ for some } m \in \mathbb{Z}\}$; thus, $B_a = B_{a+1}$, and by Theorem 2.9, $\hat{\partial}(\mathbb{L}_-^2/\mathbb{Z}) \cong \hat{\partial}_{\mathbb{Z}}(\mathbb{L}_-^2) \cong \mathbb{S}^1$, a circle.

The \mathbb{Z} -action on \mathbb{L}_-^2 extends to the boundary by $m \cdot P_a = P_{a+m}$, the typical action of the integers on the real line. The quotient by this action is thus a circle; we have the simple result that $\hat{\partial}(\mathbb{L}_-^2/\mathbb{Z}) \cong \hat{\partial}(\mathbb{L}_-^2)/\mathbb{Z}$. (The same topology arises from using $\hat{\cdot}$ -topology on $\widehat{\mathbb{L}_-^2}/\mathbb{Z}$.) It appears that the crucial point is that both \mathbb{L}_-^2 and its quotient by \mathbb{Z} have spacelike boundaries—unlike \mathbb{L}^2 , which has a null boundary.

The significance of spacelike boundaries is that the issue of continuous extension of a map to the boundary is considerably simplified if the boundary is spacelike. In brief: Spacelike boundaries guarantee continuous extension, but continuous extension can fail on timelike boundaries (or even null boundaries—even in Minkowski space), as is shown in [H2].

The details run thus: Call a point x in a past-regular chronological set X *inobservable* if $I^-(x)$ is not contained in any other IP; in effect, this says that no observer observes all the events that constitute the past of the event x , except for an observer who actually experiences x . Then X is said to have *only spacelike future boundaries* if all elements of $\hat{\partial}(X)$ are inobservable, and if the set of all inobservables is a closed subset of \hat{X} in the $\hat{\cdot}$ -topology. If X is a strongly causal spacetime M , the latter condition follows from the first, since in a spacetime proper, no points are inobservable—there's always a point to the future—and with strong causality, $\hat{\partial}(M)$ is necessarily closed in \hat{M} (Proposition 2.6 in [H2]). A future-continuous map $f : X \rightarrow Y$ is said to *preserve spacelike future boundaries* if $\hat{f} : \hat{X} \rightarrow \hat{Y}$ preserves inobservables (*i.e.*, $\hat{f}(p)$ is inobservable in \hat{Y} if p is inobservable in \hat{X}); for $f : M \rightarrow N$ a map between strongly causal spacetimes, both of which have only spacelike future boundaries, this just amounts to saying that \hat{f} takes boundary points to boundary points. Then Proposition 2.7 of [H2] says that if $f : X \rightarrow Y$ is a future-continuous map between past-regular chronological sets having only spacelike future boundaries, such that f preserves spacelike boundaries and is continuous in the $\hat{\cdot}$ -topologies, then $\hat{f} : \hat{X} \rightarrow \hat{Y}$ is also continuous in the $\hat{\cdot}$ -topologies.

For spacetimes: If M and N are strongly causal spacetimes, each satisfying the property that the entire history of no endless observer is available to any other observer, and if $f : M \rightarrow N$ is a continuous map preserving the chronology relation such that for any future-endless timelike curve σ in M , $f \circ \sigma$ is future-endless in N , then $\hat{f} : \hat{M} \rightarrow \hat{N}$ is also continuous (in the $\hat{\cdot}$ -topologies). If the hypotheses are not met, this goes wrong even in simple cases: [H2] provides a counterexample with $f : M \rightarrow \mathbb{L}^2$, where M is all points to one side of a timelike line in \mathbb{L}^2 (f is continuous, but \hat{f} is not); and this can be modified to $f' : M' \rightarrow \mathbb{L}^2$ with M' being all points to one side of a null line (map M' to M , preserving chronology, by mapping null lines onto themselves; then apply f).

It would thus seem that spacelike boundaries make for simplified topological behavior when extending actions to boundaries, and we may thus expect simple behavior in the quotients, that the boundary of the quotient is the quotient of the boundary. With spacelike future boundaries for a spacetime V and its quotient V/G , there is no difference between the quotient topology on \hat{V}/G and the $\hat{\cdot}$ -topology on that chronological set (see Proposition 3.3). Then the following seems to be likely:

Conjecture. *Let V be a spacetime with a free, properly discontinuous, and chrono-*

logical action from a group G , with $M = V/G$ being strongly causal, such that both V and M have only spacelike future boundaries. Then \widehat{M} is homeomorphic to the quotient space \widehat{V}/G , and $\hat{\partial}(M)$ is homeomorphic to the quotient space $\hat{\partial}(V)/G$ (with $\hat{\cdot}$ -topologies used on \widehat{M} and \widehat{V}); in other words, $\widehat{V}/G \cong \widehat{V}/G$ and $\hat{\partial}(V/G) \cong \hat{\partial}(V)/G$.

However, to obtain the conclusions in this conjecture, one must also know that \widehat{V}/G , as a chronological set, is past-distinguishing (see Theorem 3.4). This is a rather unnatural kind of hypothesis, in that it does not have any simple explication in terms of either of the two spacetimes proper, V and M ; it would certainly be desirable to do without it. Is it a necessary assumption, or does it follow from the other hypotheses? This is unclear at the present; no examples have come to light as yet showing this assumption to be necessary. We will see that at least for a fairly largish class of spacetimes—what are sometimes called multi-warped spacetimes—it is only the spacelike boundaries that are, indeed, the crucial ingredient, without need for an explicit additional assumption on \widehat{V}/G .

The first step in approaching the study of these quotients is to show that there is a group action on the boundary, extending that on the spacetime. This holds quite generally, on chronological sets irrespective of the nature of the boundary:

Lemma 3.1. *Let X be a past-regular chronological set with a chronological action from a group G . The G -action on X extends to a chronological G -action on \widehat{X} , continuous in the $\hat{\cdot}$ -topology.*

Proof. For each $g \in G$. Let $\phi_g : X \rightarrow X$ be defined by $\phi_g : x \mapsto g \cdot x$. In order to be able to define $\hat{\phi}_g : \widehat{X} \rightarrow \widehat{X}$, we need to show (just as in the proof of Proposition 2.6 above) that for any future chain c in X generating an element $P \in \hat{\partial}(X)$, $\phi_g[c]$ does not have two different future limits in X . But since ϕ_g is an isomorphism of the chronology relation, this is clear: Since c has no future limit in X , neither can $\phi_g[c]$. This allows the definition of $\hat{\phi}_g$; for $P \in \hat{\partial}(X)$, $\hat{\phi}_g(P) = g \cdot P$, so $\hat{\phi}_g \circ \hat{\phi}_h = \hat{\phi}_{gh}$.

Note that $\hat{\phi}_g$ is also an isomorphism of the chronology relation: If $x \ll P$ for $x \in X$ and $P \in \hat{\partial}(X)$, i.e., $x \in P$, then $g \cdot x \in g \cdot P$, so $\hat{\phi}_g(x) \ll \hat{\phi}_g(P)$; if $P \ll x$, i.e., $P \subset I^-(w)$ for some $w \ll x$, then $g \cdot P \subset g \cdot I^-(w) = I^-(g \cdot w)$ and $g \cdot w \ll g \cdot x$, so $\hat{\phi}_g(P) \ll \hat{\phi}_g(x)$; and if $P \ll Q$ ($Q \in \hat{\partial}(X)$), i.e., $P \subset I^-(w)$ for some $w \in Q$, then $g \cdot P \subset I^-(g \cdot w)$ and $g \cdot w \in g \cdot Q$, so $\hat{\phi}_g(P) \ll \hat{\phi}_g(Q)$. It follows that $\hat{\phi}_g$ is future-continuous and also a homeomorphism in the $\hat{\cdot}$ -topology.

To show the action is chronological on \widehat{X} , consider the possibility that $P \ll g \cdot P$ for $g \neq e$. This would mean that for some $w \in g \cdot P$, $P \subset I^-(w)$. Then $g^{-1} \cdot w \in P$, so $g^{-1} \cdot w \in I^-(w)$, i.e., $g^{-1} \cdot w \ll w$; but that violates the action of G on X being chronological. \square

It is worth noting that even in the simplest of cases, the extension of a free G -action to the boundary need not be free on the boundary. The example of \mathbb{L}^2 at the beginning of this section shows how the free \mathbb{Z} action on a spacetime can fail to be free on the boundary (i^+ is a fix-point of the action on $\widehat{\mathbb{L}^2}$). For an example with spacelike boundaries, consider a standard static spacetime on a sphere: the metric product $V = \mathbb{L}^1 \times \mathbb{S}^n$. Let G be a typical free group action on the sphere, such as \mathbb{Z}_2 acting (in any dimension) via $p \mapsto -p$, or \mathbb{Z}_4 acting on \mathbb{S}^3 as multiplication by i (i.e., $(x, y, z, w) \mapsto (-y, x, -w, z)$); then G acts chronologically (and freely and properly discontinuously) on V via $g \cdot (t, p) = (t, g \cdot p)$. The quotient $M = V/G$ is

also a standard static spacetime, $M = \mathbb{L}^1 \times (\mathbb{S}^n/G)$. By the results in [H3], since the Riemannian factors in both spacetimes are compact, the future chronological boundaries for both V and M are trivial, a single point: i^+ , the entire spacetime; clearly, V and M have only spacelike boundaries. The G action on $\hat{\partial}(V) = \{i^+\}$ is the degenerate action: For all $g \in G$, $g \cdot i^+ = i^+$.

The second step is to show that the projection onto the quotient defined by the group action extends to the boundaries; this requires strongly causal spacetimes and, for continuity, spacelike boundaries.

Proposition 3.2. *Let V be a spacetime with a free, properly discontinuous, and chronological action from a group G , with $M = V/G$ being strongly causal.*

a) *The projection $\pi : V \rightarrow M$ extends to a future-continuous map $\hat{\pi} : \widehat{V} \rightarrow \widehat{M}$, with $\hat{\pi}$ mapping $\hat{\partial}(V)$ onto $\hat{\partial}(M)$.*

b) *If V and M have only spacelike future boundaries, then $\hat{\pi}$ is continuous in the respective $\hat{\cdot}$ -topologies.*

Proof. As in the proof of Proposition 2.6, with no assumptions being made of past-distinguishing or past-determination, we need to show that for any future chain c in V , $\pi[c]$ cannot have two future limits in M , in order that $\hat{\pi}$ can be defined; and, once it is known to exist, we must check separately that $\hat{\pi}$ preserves the chronology relation.

By Proposition 2.2 of [H2], a future limit of a future chain is the same as a limit of the sequence in the $\hat{\cdot}$ -topology; and by Theorem 2.3 of [H2], for a strongly causal spacetime the $\hat{\cdot}$ -topology is the same as the manifold topology. Thus, what's needed for the existence of $\hat{\pi}$ is to show that for any future-endless timelike curve c in V , $\pi \circ c$ cannot have two future endpoints in M ; but this is just a consequence of the manifold topology being Hausdorff.

In fact, if c generates an element of $\hat{\partial}(V)$, $\pi \circ c$ cannot have any limit in M (this will be useful for part (b)). For suppose $\pi \circ c$ has $p \in M$ as a limit. Let U be a neighborhood of p such that $\pi^{-1}[U]$ is a set of components $\{W_g \mid g \in G\}$ evenly covering U , i.e., π is a homeomorphism from each W_g to U ; let x_g be the pre-image of p in W_g . For U sufficiently small, $\pi \circ c$ enters U and does not leave it; consequently, c cannot exit any W_g , so it enters exactly one W_h and never leaves it. Then, since $\pi : W_h \rightarrow U$ is a homeomorphism, c must have x_h as a limit, violating the fact that c is future-endless.

This allows for the definition of $\hat{\pi} : \widehat{V} \rightarrow \widehat{M}$ in such a way that boundary points are mapped to boundary points: For a future-endless curve c generating $P \in \hat{\partial}(V)$, $\hat{\pi}(P) = I^-[\pi \circ c] = \{\pi(x) \mid x \ll_G c(t) \text{ for some } t\} = \{\pi(x) \mid x \ll c(t) \text{ for some } t\} = \pi[P]$. As in Proposition 2.6, to show $\hat{\pi}$ is future-continuous, we need only show that it is chronological:

Consider $x, y \in V$ and $P, Q \in \hat{\partial}(V)$. If $x \ll y$, then $\hat{\pi}(x) = \pi(x) \ll \pi(y) = \hat{\pi}(y)$. If $x \ll P$, i.e., $x \in P$, then $\hat{\pi}(x) = \pi(x) \in \pi[P] = \hat{\pi}(P)$, i.e., $\hat{\pi}(x) \ll \hat{\pi}(P)$. If $P \ll y$, i.e., $P \subset I^-(w)$ for some $w \ll y$, then $\hat{\pi}(P) = \pi[P] \subset I^-(\pi(w))$ and $\pi(w) \ll \pi(y) = \hat{\pi}(y)$, i.e., $\hat{\pi}(P) \ll \hat{\pi}(y)$. If $P \ll Q$, i.e., $P \subset I^-(w)$ for some $w \in Q$, then $\hat{\pi}(P) = \pi[P] \subset I^-(\pi(w))$ and $\pi(w) \in \pi[Q] = \hat{\pi}(Q)$, i.e., $\hat{\pi}(P) \ll \hat{\pi}(Q)$. Thus, $\hat{\pi}$ is chronological, and part (a) is established.

For part (b), we will use Proposition 2.7 of [H2]. We are now assuming that both V and M have only spacelike future boundaries; since they are manifolds, they are automatically past-regular. By Proposition 2.1, $\pi : V \rightarrow M$ is future-continuous.

Then the only remaining fact we need is that $\hat{\pi} : \hat{V} \rightarrow \hat{M}$ preserve inobservables; then Proposition 2.7 of [H2] will tell us that $\hat{\pi}$ is continuous in the $\hat{\cdot}$ -topologies.

Since V is a spacetime, the only inobservables in \hat{V} are the boundary elements (all of which are inobservable, since V has only spacelike future boundaries). If $P \in \hat{\partial}(V)$ is generated by the future-endless timelike curve c , then $\hat{\pi}(P)$ is generated by $\pi \circ c$. All we need is that $\hat{\pi}(P)$ is inobservable, *i.e.*, not contained in any other IP. We know $\hat{\pi}(P) = I^-[\pi \circ c]$. Since c is future-endless, so is $\pi \circ c$ (as shown above). That means there is no $q \in M$ so that $I^-(q) = \pi[P] = \hat{\pi}(P)$; in other words, $\hat{\pi}(P) \in \hat{\partial}(M)$. But since M is assumed to have only spacelike future boundaries, all elements of $\hat{\partial}(M)$ are inobservable. \square

The hypotheses are really needed for Proposition 3.2. As an example of what can go wrong for part (a) if M is not strongly causal, let $V = \mathbb{L}^2$, and let the integers \mathbb{Z} act on \mathbb{L}^2 in a null fashion: $n \cdot (x, t) = (x + n, t + n)$. Note that the quotient $M = \mathbb{L}^2/\mathbb{Z}$ is not past-distinguishing (as well as not strongly causal): Let $[x, t]$ denote the equivalence class of (x, t) ; then $I^-([x, t]) = \{[y, s] \mid s - y < t - x\}$, so that $I^-([x, t]) = I^-([x + a, t + a])$ for all a , a circle's worth of points in M , all with the same past. Consider the boundary IP $P = \{(x, t) \mid t < x\}$; it is generated by the future chain $c(n) = (n, n - \frac{1}{n})$. Examine $\pi[c] = \{[n, n - \frac{1}{n}]\} = \{[0, -\frac{1}{n}]\}$: $I^-[\pi[c]]$ is not a boundary IP, since it is the same as $I^-([0, 0])$; but that is the same as any $I^-([a, a])$. Thus, there is no way to define $\hat{\pi}(P)$, as it is supposed to be each $[a, a]$ (*i.e.*, each $[a, a]$ is a future limit of $\pi[c]$).

An example of what can go wrong for part (b) if the spacetimes have timelike boundaries will be given at the end of this section (it is rather complicated but worth knowing about).

As was mentioned before, we really have two topologies to consider for \hat{V}/G (hence, for $\hat{\partial}(V)/G$): With the $\hat{\cdot}$ -topology on \hat{V} , we can consider the quotient topology on \hat{V}/G . Alternatively, since \hat{V} is a past-regular chronological set with a chronological G -action, \hat{V}/G is also (by Proposition 2.7) a past-regular chronological set, yielding a $\hat{\cdot}$ -topology. As the example of \mathbb{L}^2 at the beginning of this section showed, these topologies need not be the same in general; but with spacelike boundaries, they are.

Proposition 3.3. *Let V be a spacetime with a free, properly discontinuous, and chronological action from a group G , with $M = V/G$ being strongly causal. Suppose both V and M have only spacelike future boundaries; then the quotient topology and $\hat{\cdot}$ -topology on \hat{V}/G are the identical, *i.e.*, $(\hat{V}/G, \text{quotient topology}) \cong (\hat{V}/G, \hat{\cdot}\text{-topology})$.*

Proof. For any topological space A with a G -action, the quotient map $\pi_A : A \rightarrow A/G$ is continuous (with the quotient topology on A/G) and has the following universal property: If $f : A \rightarrow B$ is any continuous G -invariant map, then there is a unique continuous map $f_G : A/G \rightarrow B$ such that $f_G \circ \pi_A = f$; f_G is defined by $f_G : \langle a \rangle \mapsto f(a)$, where $\langle a \rangle$ denotes the G -equivalence class of a .

Since V is past-regular, so is \hat{V} , hence also \hat{V}/G (Proposition 2.7). Consider the quotient map $\Pi : \hat{V} \rightarrow \hat{V}/G$ as a function between chronological sets. Note that this is not the map $\hat{\pi} = \hat{\pi}_{\hat{V}}$ from Proposition 3.2. On the set-level, this is the same as the universal map $\pi_{\hat{V}}$; but we will use Π instead of $\pi_{\hat{V}}$ to denote that we want

to use the $\hat{\cdot}$ -topology on the target space, which may (*a priori*) be different from the quotient topology there.

We need to see that Π is continuous. We know that V/G is an open subset of \hat{V}/G and that on that open set, the $\hat{\cdot}$ -topology from \hat{V}/G is the same as the $\hat{\cdot}$ -topology on V/G as a chronological set in its own right (Theorem 2.4 in [H2]); and we know that that is the same as the manifold topology on $V/G = M$. We also know that Π takes boundary points to boundary points, so that it restricts to $\pi_V : V \rightarrow M$. That map is continuous; thus, all we need be concerned about is the boundary in \hat{V} .

So consider a sequence of boundary elements $\sigma = \{P_n\}$ in \hat{V} . The boundary is closed, so the only possible members of $L(\sigma)$ are boundary elements also. Suppose $Q \in L(\sigma)$; we need to show that $\Pi(Q) \in L(\Pi[\sigma])$. Let $\langle \cdot \rangle$ denote equivalence class under the G -action. Since V/G has only spacelike future boundaries, all we need do is consider if for all $\langle x \rangle \ll \Pi(Q) = \langle Q \rangle$, eventually $\langle x \rangle \ll \Pi(P_n) = \langle P_n \rangle$. But $\langle x \rangle \ll \langle Q \rangle$ if and only if $x \ll_G Q$, *i.e.*, if for some $g \in G$, $g \cdot x \in Q$. If this obtains, then, as $Q \in L(\{P_n\})$, eventually $g \cdot x \in P_n$; and that yields $\langle x \rangle \ll \langle P_n \rangle$. Hence, Π is continuous.

With $\Pi : \hat{V} \rightarrow (\hat{V}/G, \hat{\cdot}\text{-topology})$ continuous, the universal property of $\pi_{\hat{V}}$ yields for us the continuous map $\Pi_G : (\hat{V}/G, \text{quotient topology}) \rightarrow (\hat{V}/G, \hat{\cdot}\text{-topology})$; since Π is just $\pi_{\hat{V}}$ on the set-level, Π_G is just the identity map on the set-level. We need to show this map is proper, *i.e.*, that limits in the target space correspond to limits in the domain. Since both topologies yield the manifold topology on $V/G = M$, we need only look at the boundary-points for limits.

So consider $Q \in \hat{\partial}(V)$ and a sequence $\sigma = \{\alpha_n\}$ in \hat{V} (each α_n either in V or in $\hat{\partial}(V)$, it doesn't matter) with $\langle Q \rangle \in L(\{\langle \alpha_n \rangle\})$; we need to see that $\langle Q \rangle$ is a limit point of $\langle \sigma \rangle = \pi_V[\sigma]$ in the quotient topology. Let Q be generated by a future chain $\{y_m\}$. For all m , $\langle y_m \rangle \ll \langle Q \rangle$, so $\langle y_m \rangle \ll \langle \alpha_n \rangle$ for n sufficiently large, *i.e.*, there is some number k_m and for all $n \geq k_m$ there is some $g_m^n \in G$ such that $y_m \ll g_m^n \cdot \alpha_n$. Since $\{y_m\}$ is a future chain, this means that for all m , for all $i \leq m$, for all $n \geq k_m$, $y_i \ll g_m^n \cdot \alpha_n$. We can turn this around: For all i , for all $m \geq i$, for all $n \geq k_m$, $y_i \ll g_m^n \cdot \alpha_n$; in particular (letting $n = k_m$), for all i , for all $m \geq i$, $y_i \ll g_m^{k_m} \cdot \alpha_{k_m}$. Let $\bar{\alpha}_m = g_m^{k_m} \cdot \alpha_{k_m}$; then we've just shown that for all i , $y_i \ll \bar{\alpha}_m$ for all $m \geq i$. Since V has only spacelike future boundaries, this is sufficient to show that $Q \in L(\bar{\sigma})$, where $\bar{\sigma} = \{\bar{\alpha}_m\}$. Then, since $\pi_{\hat{V}}$ is continuous, $\langle Q \rangle$ is a limit of the sequence $\langle \bar{\sigma} \rangle = \pi_{\hat{V}}[\bar{\sigma}]$.

That's not quite what we wanted; we need $\langle Q \rangle$ a limit of $\langle \sigma \rangle$. But it's actually good enough: For if we consider any subsequence $\tau \subset \sigma$, then the same procedure will produce a subsubsequence $\bar{\tau} \subset \tau$ with $\langle Q \rangle$ a limit point of $\langle \bar{\tau} \rangle$. With $\langle Q \rangle$ a limit point of some subsubsequence of every subsequence of $\langle \sigma \rangle$, we conclude $\langle Q \rangle$ is a limit of $\langle \sigma \rangle$. \square

The last step is to show that the projection from \hat{V} to \hat{M} induces a homeomorphism between the quotient space \hat{V}/G and \hat{M} ; this requires all the hypotheses of Proposition 3.2(b) and, it seems, one additional hypothesis. It is easy enough to show there is a continuous map between these spaces; the trick is to show it is injective. This can be accomplished by requiring that \hat{V}/G be past-distinguishing; but it is unclear whether this is a truly necessary hypothesis.

Theorem 3.4. *Let V be a spacetime with a free, properly discontinuous, and*

chronological action from a group G , with $M = V/G$ being strongly causal, such that both V and M have only spacelike future boundaries. Suppose that \widehat{V}/G is past-distinguishing. Then \widehat{M} is homeomorphic to the quotient space \widehat{V}/G , and $\hat{\partial}(M)$ is homeomorphic to $\hat{\partial}(V)/G$ (with $\widehat{\cdot}$ -topologies used on \widehat{M} and \widehat{V}).

Proof. Let $\pi : V \rightarrow M$ be projection. By Proposition 3.2, π extends to the future-completions $\hat{\pi} : \widehat{V} \rightarrow \widehat{M}$, and this is continuous in the respective $\widehat{\cdot}$ -topologies. Clearly, π is G -invariant, so $\hat{\pi}$ is also G -invariant when restricted to V . As shown in the proof of Proposition 3.2, the action of $\hat{\pi}$ on boundary elements is simple for $P \in \hat{\partial}(V)$: $\hat{\pi}(P) = \pi[P]$. Thus, $\hat{\pi}$ is also G -invariant on $\hat{\partial}(V)$: $\hat{\pi}(g \cdot P) = \pi[g \cdot P] = \pi[P] = \hat{\pi}(P)$. It follows that there is an induced map $\hat{\pi}_G : \widehat{V}/G \rightarrow \widehat{M}$ (so that with $\pi_{\widehat{V}} : \widehat{V} \rightarrow \widehat{V}/G$ as in the proof of Proposition 3.3, $\hat{\pi}_G \circ \pi_{\widehat{V}} = \hat{\pi}$), which is continuous using the quotient topology on \widehat{V}/G . By virtue of Proposition 3.3, that is the same as the $\widehat{\cdot}$ -topology on \widehat{V}/G .

Note that the G -action on \widehat{V} preserves $\hat{\partial}(V)$: For $x \in V$, $g \cdot x \in V$ also. Thus, \widehat{V}/G can be broken into two parts: $\widehat{V}/G = [V/G] \cup [\hat{\partial}(V)/G] = M \cup [\hat{\partial}(V)/G]$. Note that $\hat{\pi} : V \cup \hat{\partial}(V) \rightarrow M \cup \hat{\partial}(M)$ neatly splits as $\pi : V \rightarrow M$ and a map we can call $\pi^\partial : \hat{\partial}(V) \rightarrow \hat{\partial}(M)$ (this was contained in the proof of Proposition 3.2(b)). It follows that $\hat{\pi}_G : \widehat{V}/G \rightarrow \widehat{M}$ splits as $\text{id}_M : M \rightarrow M$ and a map $\hat{\pi}_G^\partial : \hat{\partial}(V)/G \rightarrow \hat{\partial}(M)$.

Note that id_M really is the identity on M in the topological category: By Theorem 2.4 of [H2], the topology on a spacetime N induced, as a topological subspace, by the $\widehat{\cdot}$ -topology on \widehat{N} is the same as the $\widehat{\cdot}$ -topology on N as a chronological set; that latter topology, by Theorem 2.3 of [H2], is the same as the manifold topology if N is strongly causal; and (again, for N being strongly causal) by Proposition 2.6 of [H2], $\hat{\partial}(N)$ is closed in \widehat{N} in the $\widehat{\cdot}$ -topology. It follows that the $\widehat{\cdot}$ -topology on \widehat{V} induces the manifold topology on V (note that Proposition 1.3 tells us V is strongly causal), and that the quotient topology on \widehat{V}/G induces the manifold topology on the open subset $V/G = M$.

So we need to concentrate on the nature of $\hat{\pi}_G^\partial : \hat{\partial}(V)/G \rightarrow \hat{\partial}(M)$, which is defined thus: Let P be in $\hat{\partial}(V)$, and let $\langle P \rangle$ denote the equivalence class of P under the G -action on $\hat{\partial}(V)$ (and also use $\langle x \rangle$ for the equivalence class of $x \in V$); then $\hat{\pi}_G^\partial(\langle P \rangle) = \pi^\partial(P) = \pi[P]$. The most important aspect to note is that this is injective (that is what typically fails to be the case with non-spacelike boundaries). Injectivity of $\hat{\pi}_G^\partial$ comes directly from the assumption that \widehat{V}/G is past-distinguishing:

Consider $P \in \hat{\partial}(V)$ and its image $\langle P \rangle$ in \widehat{V}/G . What is the past of $\langle P \rangle$ in that chronological set? For $x \in V$, $\langle x \rangle \ll \langle P \rangle$ if and only if in \widehat{V} , $x \ll_G P$, i.e., for some $g \in G$, $x \ll g \cdot P$, i.e., for some g , $x \in g \cdot P$, i.e., $x \in \bigcup \langle P \rangle$ (thinking of $\langle P \rangle$ as a collection of subsets of V); and that last is equivalent to $\langle x \rangle$ being among $\{\langle y \rangle = \pi(y) \mid y \in P\}$. It follows that $\langle x \rangle \ll \langle P \rangle$ if and only if $\langle x \rangle \in \pi[P] = \hat{\pi}_G^\partial(\langle P \rangle)$. What about other elements of \widehat{V}/G possibly being in the past of $\langle P \rangle$? For $Q \in \hat{\partial}(V)$, $\langle Q \rangle \ll \langle P \rangle$ if and only if for some $g \in G$, $Q \ll g \cdot P$, i.e., for some $w \in g \cdot P$, $Q \subset I^-(w)$; but that is impossible since V has only spacelike future boundaries. Therefore, we can conclude that $\hat{\pi}_G^\partial(\langle P \rangle) = I^-(\langle P \rangle)$; thus, $\hat{\pi}_G^\partial$ is injective precisely when \widehat{V}/G is past-distinguishing.

So we now know that $\hat{\pi}_G : \widehat{V}/G \rightarrow \widehat{M}$ is a continuous bijection. By Proposition

3.3, we know that the quotient topology on \widehat{V}/G is the same as the $\widehat{\cdot}$ -topology using the chronological structure on \widehat{V}/G . Thus, we can show $\hat{\pi}_G$ is a homeomorphism by showing that it is an isomorphism of chronological sets. First consider x and y in V : There is no distinction between their images in \widehat{V}/G and in \widehat{M} (i.e., $\langle x \rangle = \pi(x)$) and the chronology relation in either case resolves to $x \ll_G y$. Next consider also P and Q in $\partial(V)$. As shown above, we have $\langle x \rangle \ll \langle P \rangle$ if and only if $\langle x \rangle \in \pi[P]$; but $\langle x \rangle = \hat{\pi}_G(\langle x \rangle)$ and $\pi[P] = \hat{\pi}_G(\langle P \rangle)$, so this is equivalent to $\hat{\pi}_G(\langle x \rangle) \in \hat{\pi}_G(\langle P \rangle)$, and in \widehat{M} that is the same as $\hat{\pi}_G(\langle x \rangle) \ll \hat{\pi}_G(\langle P \rangle)$. Finally, we cannot have either $\langle P \rangle \ll \langle x \rangle$ or $\langle P \rangle \ll \langle Q \rangle$, as that would mean $P \ll_G x$ and $P \ll_G Q$ respectively, resulting in $P \ll g \cdot x$ and $P \ll g \cdot Q$, impossible for V having only spacelike future boundaries; and as M has only spacelike future boundaries, we likewise cannot have $\hat{\pi}_G(\langle P \rangle) \ll \hat{\pi}_G(\langle x \rangle)$ (i.e., $\pi[P] \ll \pi(x)$) or $\hat{\pi}_G(\langle P \rangle) \ll \hat{\pi}_G(\langle Q \rangle)$ (i.e., $\pi[P] \ll \pi[Q]$). Therefore, in all cases, $\hat{\pi}_G(\alpha) \ll \hat{\pi}_G(\beta)$ if and only if $\alpha \ll \beta$.

Since $\hat{\pi}_G : \widehat{V}/G \rightarrow \widehat{M} = \widehat{V}/G$ exactly duplicates the chronology relations, it is a homeomorphism. The same is true for the restriction to the closed subsets $\hat{\pi}_G^\partial : \partial(V)/G \rightarrow \partial(M) = \partial(V/G)$. \square

Example: multi-warped spacetimes.

It is unclear whether the hypothesis in Theorem 3.4 that \widehat{V}/G be past-distinguishing is actually necessary; it seems possible that it may be a consequence of the other hypotheses. Here is a class of spacetimes in which that is true: multi-warped spacetimes (called multiply warped in [H2], but referred to as “multi-warped” in some recent publications, such as [GS]). These are spacetimes which have the form $(a, b) \times_{f_1} K_1 \cdots \times_{f_m} K_m$, where (a, b) is an interval, finite or infinite, in \mathbb{L}^1 , each K_i is a Riemannian manifold with metric h_i , each $f_i : (a, b) \rightarrow \mathbb{R}$ is a positive function, and the spacetime metric is $-(dt)^2 + f_1(t)h_1 + \cdots + f_m(t)h_m$. These include inner Schwarzschild, Robertson-Walker spaces, the Kasner spacetimes, and many others.

It was shown in [H2] (Proposition 5.2) that if M is a multi-warped spacetime such that for each i , (K_i, h_i) is complete and for some finite $b^- < b$, $\int_{b^-}^b f_i^{-1/2} < \infty$, then $\partial(M)$ is spacelike and is homeomorphic to $K = K_1 \times \cdots \times K_m$, with \widehat{M} homeomorphic to $(a, b] \times K$ and $\partial(M)$ appearing there as $\{b\} \times K$. Before examining the behavior of multi-warped spacetimes for Theorem 3.4, it is perhaps well to expand the conditions under which it can be known that $\partial(M)$ is spacelike. This is somewhat lengthy, but fairly straight-forward.

Proposition 3.5. *Let $M = (a, b) \times_{f_1} K_1 \cdots \times_{f_m} K_m$ be a multi-warped spacetime. Let the Riemannian factors be arranged such that (for some finite $b^- < b$) the first k warping functions— i from 1 to k —obey $\int_{b^-}^b f_i^{-1/2} < \infty$, and the rest— i from $k+1$ to m —obey $\int_{b^-}^b f_i^{-1/2} = \infty$. Then the following hold:*

- (a) *If some Riemannian factor K_i is incomplete, then $\partial(M)$ has timelike-related elements.*
- (b) *If for some $i \geq k+1$, K_i is not compact, then $\partial(M)$ has null-related elements, i.e., $P \subset P'$ with $P \neq P'$.*
- (c) *If neither of those occur, then M has only spacelike future boundaries.*

In the last case, $\partial(M)$ is homeomorphic to $K^0 = K_1 \times \cdots \times K_k$. Furthermore, \widehat{M} is homeomorphic to $((a, b] \times K^0 \times K') / \sim$, where $K' = K_{k+1} \times \cdots \times K_m$ and \sim is the equivalence relation defined by $(b, x^0, x') \sim (b, x^0, y')$ for any $x^0 \in K^0$ and

$x', y' \in K'$; $\hat{\partial}(M)$ appears there as $\{b\} \times K^0 \times \{*\}$.

Proof. For part (a), suppose K_i is incomplete. Let $M_i = (a, b) \times_{f_i} K_i$; then M_i is conformal to the product static spacetime $\bar{M}_i = (a', b') \times K_i$, where $b' = d + \int_d^b f_i^{-1/2}$ and $a' = d - \int_a^d f_i^{-1/2}$ for some choice of d between a and b . It is easy to find timelike related elements $P_i \ll P'_i$ in $\hat{\partial}(\bar{M}_i)$, such as by using an endless unit-speed geodesic c_i of finite length in K_i and letting P_i and P'_i be generated, respectively, by the null geodesics $\sigma_i(t) = (t_0 + t, c_i(t))$ and $\sigma'_i(t) = (t'_0 + t, c_i(t))$ with $t'_0 > t_0$; the numbers t_0 and t'_0 have to be chosen so that, with whatever the domain of c_i is, σ_i and σ'_i both remain within the comprehension of \bar{M}_i , but this is not difficult. We may consider σ_i and σ'_i to be curves in M_i as well as in \bar{M}_i , though their first coordinates will be parametrized differently: $\sigma_i(t) = (\tau(t), c_i(t))$ and $\sigma'_i(t) = (\tau'(t), c_i(t))$. Note that although σ_i and σ'_i are likely not to be geodesics in M_i , they are still null curves in M_i ; the sets P_i and P'_i are still IPs in M_i generated by σ_i and σ'_i , they are elements in the future chronological boundary of M_i , and they still obey $P_i \ll P'_i$ in \widehat{M}_i : There is some $(s, x_i) \in M_i$ such that $P_i \subset I^-((s, x_i))$ and $(s, x_i) \in P'_i$.

Now pick points $x_j \in K_j$ for each $j \neq i$, and let c be the curve in $K = K_1 \times \cdots \times K_m$ defined by the projection to K_i being c_i and the projection to each K_j being constant at x_j for $j \neq i$. Let σ and σ' be curves in M defined by $\sigma(t) = (\tau(t), c(t))$ and $\sigma'(t) = (\tau'(t), c(t))$; these are clearly endless null curves. Let P and P' be the IPs in M generated by σ and σ' respectively; they are boundary elements. Let $x = (x_1, \dots, x_m)$ (so that this is x_i in the projection to K_i). Note that as the past in M_i of $(s + \delta, x_i)$ (for $\delta > 0$ but small) includes σ_i , it follows that the past in M of $(s + \delta, x)$ includes σ ; hence, $P \subset I^-((s + \delta, x))$. Similarly, as $(s + \delta, x_i)$ is in the past of σ'_i , it follows that $(s + \delta, x)$ is in the past of σ' , i.e., $(s + \delta, x) \in P'$. Therefore, $P \ll P'$.

For part (b), suppose K_i is not compact and $\int_{b-}^b f_i^{-1/2} = \infty$. If K_i is not complete, then we are done, by part (a); so we may assume K_i is complete. Any complete, non-compact Riemannian manifold contains a ray—a geodesic minimizing over all intervals—of infinite length (consider any sequence of points $\{p_n\}$ with distances from p_0 going to infinity and minimizing unit-speed geodesics γ_n from p_0 to p_n ; the tangent vectors $\{\dot{\gamma}_n(0)\}$ have an accumulation vector v , and that generates the requisite ray). Let c_i be this ray in K_i . As in part (a), let $M_i = (a, b) \times_{f_i} K_i$ and let \bar{M}_i be the conformally related product static spacetime $(a', b') \times K_i$; but in this case, we know $b' = \infty$. As before, let σ_i and σ'_i be the null geodesics $\sigma_i(t) = (t_0, c_i(t))$ and $\sigma'_i(t) = (t'_0, c_i(t))$ with $t'_0 > t_0$, generating IPs P_i and P'_i respectively. The results of [H3] on the future chronological boundary of standard static spacetimes apply in this case, even if a' is finite: The Busemann function for c_i is finite, since c_i is a ray; thus, P_i and P'_i are distinct elements in $\hat{\partial}(M_i)$, with $P_i \subset P'_i$. (The material in [H3] shows that elements of $\hat{\partial}(\mathbb{L}^1 \times N)$ are generated by functions of the form $b_c(p) = \lim_{t \rightarrow \infty} (t - d(p, c(t)))$ for c a curve in N of no more than unit speed; that is the Busemann function for c .)

We proceed as before, selecting points $x_j \in K_j$ for $j \neq i$ and using them to define σ and σ' as endless null curves in M , generating boundary IPs P and P' . As σ_i lies in the past of σ'_i , so does σ lie in the past of σ' ; hence, $P \subset P'$. As there exists some point (s, x_i) in P'_i but not in P_i , so does (s, x) lie in P' but not in P , where x has projections to x_j in K_j .

For part (c), we emulate the proof for the case $k = m$ given in Proposition

5.2 of [H2]. First let $\bar{M} = ((a, b] \times K^0 \times K') / \sim$, with $(b, x^0, x') \sim (b, x^0, y')$ for any $x^0 \in K^0$ and any $x', y' \in K'$. Let $[x^0]$ denote the equivalence class of (b, x^0, x') . Define a chronology relation on \bar{M} , extending that on M , as follows: For $(t, x^0, x') \in M$ and $y^0 \in K^0$, set $(t, x^0, x') \ll [y^0]$ if and only if there is a timelike curve in $M^0 = (a, b) \times_{f_1} K_1 \cdots \times_{f_k} K_k$ starting at (t, x^0) and approaching (b, y^0) . This amounts to saying there is a continuous curve $c^0 : [t, b] \rightarrow K^0$, going from x^0 to y^0 , differentiable on $[t, b)$, with $\sum_{i=1}^k f_i(s)^{-1/2} |\dot{c}_i^0|_i < 1$ for all $s \in [t, b)$, where c_i^0 denotes the projection of c^0 to K_i and $|\cdot|_i$ is the norm in the Riemannian metric on K_i . Let \bar{I}^+ and \bar{I}^- denote the future and past operators in the chronological set (\bar{M}, \ll) .

This amounts to the same construction used in Proposition 5.2 of [H2], applied to the spacetime M^0 , save that in that case, the chronological relation was defined on $\bar{M}^0 = (a, b] \times K^0$ instead of $\bar{M} = ((a, b) \times K^0 \times K') / \sim$. More precisely: $(t, x^0, x') \ll [y^0]$ in \bar{M} if and only if $(t, x^0) \ll (b, y^0)$ in \bar{M}^0 . The Lemma of Proposition 5.2 assures us that for any $(t, x^0) \in M^0$, $\bar{I}^+((t, x^0))$ is open in \bar{M}^0 , *i.e.*, the set of $y^0 \in K^0$ such that $(b, y^0) \gg (t, x^0)$ is open. It follows that $\bar{I}^+(t, x^0, x')$ is open in \bar{M} .

For any $x^0 \in K^0$, define $Q_{x^0} \subset M$ to be $\bar{I}^-([x^0])$. The analogous set in M^0 is $Q_{x^0}^0$, the past of (b, x^0) ; evidently, $Q_{x^0} = Q_{x^0}^0 \times K'$. We will see that $\hat{\partial}(M) = \{Q_{x^0} \mid x^0 \in K^0\}$, just as in Proposition 5.2 of [H2] it was shown that $\hat{\partial}(M^0) = \{Q_{x^0}^0 \mid x^0 \in K^0\}$.

To see that any Q_{x^0} is an IP, pick some point $x' \in K'$, and consider the curve $\sigma(t) = (t, x^0, x')$; we need to show $Q_{x^0} = I^-[\sigma]$. Surely anything to the past of σ lies in Q_{x^0} (a timelike curve in M from (s, y^0, y') to (t, x^0, x') projects to a timelike curve in M^0 from (s, y^0) to (t, x^0)). For the converse, showing that Q_{x^0} lies in the past of σ , consider any $(s, y^0, y') \in Q_{x^0}$. We know from Proposition 5.2 $Q_{x^0}^0 = I^-[\sigma^0]$ in M^0 , where σ^0 is the curve $\sigma^0(t) = (t, x^0)$. This tells us (s, y^0) is in the past of some (t, x^0) in M^0 ; hence, $(s, y^0, y') \ll (t, x^0, y')$ in M . The trick is to show $(t, x^0, y') \ll (t', x^0, x')$ for some $t' < b$; this is where we use $\int_{b-}^b f_j^{-1/2} = \infty$ for each $j > k$.

Consider any $j > k$, and, as before, let $M_j = (a, b) \times_{f_j} K_j$. Since $\int_{b-}^b f_j^{-1/2} = \infty$, M_j is conformal to the product static spacetime $\bar{M}_j = (a', \infty) \times K_j$. For any $p, q \in K_j$, for any $\tau > a'$, $(\tau', p) \gg (\tau, q)$ so long as $\tau' > \tau + d_j(p, q)$, where d_j is the distance function from the Riemannian metric on K_j . Translating back to M_j (which has the same causal structure), we see that for any $p, q \in K_j$, for any $s \in (a, b)$, there is some $s' \in (a, b)$ with $(s', p) \gg (s, q)$. We then apply this to $s = t$, $p = x'_{k+1}$, and $q = y'_{k+1}$ to obtain a t'_1 with $(t, y'_{k+1}) \ll (t'_1, x'_{k+1})$ in M_{k+1} . This gives us, in M , $(t, x^0, y'_{k+1}, y'_{k+2}, \dots, y'_m) \ll (t'_1, x^0, x'_{k+1}, y'_{k+2}, \dots, y'_m)$; in other words, we have shown we can travel futurewards from y' towards x' , changing only in the first factor (in K_{k+1}). Repeating successively in the other factors, we end up with some $t' = t'_{m-k}$ so that $(t, x^0, y') \ll (t', x^0, x')$, as desired.

We need to know that these Q_{x^0} are the only elements of $\hat{\partial}(M)$. Let σ be any future-endless timelike curve in M ; we can assume $\sigma(t) = (t, c^0(t), c'(t))$ for curves c^0 and c' in, respectively, K^0 and K' . We need to find $x^0 \in K^0$ such that $I^-[\sigma] = Q_{x^0}$.

Consider the projection σ^0 of σ to M^0 . From Proposition 5.2 of [H2], $I^-[\sigma^0] = Q_{x^0}^0$, where x^0 is a point in K^0 that c^0 approaches. We need to see that $I^-[\sigma] = Q_{x^0}$,

i.e., $Q_{x^0}^0 \times K'$. For any point $(t, y^0, y') \ll \sigma(s)$, we also have, in M^0 , $(t, y^0) \ll \sigma^0(s)$, *i.e.*, $(t, y^0) \in Q_{x^0}^0$, so $(t, y^0, y') \in Q_{x^0}^0 \times K'$. Conversely, consider any $(t, y^0) \ll \sigma^0(s)$ (any s) and any $y' \in K'$. It will suffice to show there is some s' with $(\sigma^0(s), y') \ll (\sigma^0(s'), c'(s'))$; this is where the compactness of K_j for $j > k$ comes into play.

We need to know how points (s, z^0, z') and (r, w^0, w') in M are timelike-related; this requires a curve $\rho : [s, r] \rightarrow M$ of the form $\rho(t) = (t, e^0(t), e'(t))$ for curves $e^0 : [s, r] \rightarrow K^0$ and $e' : [s, r] \rightarrow K'$ satisfying

$$(*) \quad 1 > \sum_{i=1}^k f_i(t) |\dot{e}_i^0(t)|_i^2 + \sum_{j=k+1}^m f_j(t) |\dot{e}'_j(t)|_j^2$$

with e^0 going from z^0 to w^0 and e' going from z' to w' . Let us first ignore the K^0 factor. The crucial fact in the K' factor is this: For given s , for any $\delta > 0$, for any $z', w' \in K'$, there is some $r \in (s, b)$ and a curve $e' : [s, r] \rightarrow K'$ from z' to w' so that for each $j > k$, $f_j(t) |\dot{e}'_j(t)|_j^2 < \delta^2$ for all $t \in [s, r]$. For if we insist that $|\dot{e}'_j(t)|_j < \delta f_j(t)^{-1/2}$, then the length L_j of e'_j is bounded by $\delta \int_s^r f_j(t)^{-1/2} dt$; but since $\int_s^b f_j^{-1/2} = \infty$, this bound can be as large as we like, by suitable adjustment of r (still keeping $r < b$). Since each K_j is compact, there is some maximum length that will suffice for all L_j , $k+1 \leq j \leq m$, no matter what the points z'_j and w'_j are (just choose the maximum diameter among the K_j). Thus, some $r < b$ will accommodate all the curves e'_j on the interval $[s, r]$, going from wherever to wherever. In effect, we can entirely ignore the second term in inequality $(*)$, as with proper choice of r , a curve $e' : [s, r] \rightarrow K'$ will exist from z' to w' that makes that term $< (m-k)\delta^2$. Thus, given $s \in (a, b)$, $z^0, w^0 \in K^0$, and $z', w' \in K'$, there is some $r < b$ with $(s, z^0, z') \ll (r, w^0, w')$ in M if and only if there is some $r < b$ with $(s, z^0) \ll (r, w^0)$ in M^0 . It follows that $I^+((s, z^0, z'))$ includes all points of the form (r, w^0, w') such that $(r, w^0) \gg (s, z^0)$ and r is sufficiently large (but still $< b$).

So let us now examine the question of whether there is some s' with $(\sigma^0(s), y') \ll (\sigma^0(s'), c'(s'))$. We know now that $I^+(\sigma^0(s), y') = I^+(s, c^0(s), y')$ includes all points of the form (r, w^0, w') with $(r, w^0) \gg (s, c^0(s))$ and r sufficiently large. This includes all points of the form $(\sigma^0(s'), w') = (s', c^0(s'), w')$ for s' sufficiently large, since $\sigma^0(s') \gg \sigma^0(s)$ for $s' > s$. And in particular, this includes $(\sigma^0(s'), c'(s'))$ for s' sufficiently large. Therefore, $I^-[\sigma] = Q_{x^0}$.

Now that we know $\hat{\partial}(M) = \{Q_{x^0} \mid x^0 \in K^0\}$, we need to know that these elements are distinct for different x^0 . But this is easy, since $Q_{x^0} = Q_{x^0}^0 \times K'$ and from Proposition 5.2 in [H2], $Q_{x^0}^0 \neq Q_{y^0}^0$ for $x^0 \neq y^0$. Thus, we have fully identified $\hat{\partial}(M)$ with K^0 and, hence, \widehat{M} as the chronological set \bar{M} , with $\hat{\partial}(M)$ sitting there as $\{b\} \times K^0 \times \{*\}$. We still must verify that the topology on \widehat{M} is that claimed.

First note we can rely on Proposition 5.2 to show us that $\hat{\partial}(M)$ is spacelike: If $Q_{x^0}^0 \times K' \subset Q_{y^0}^0 \times K'$, then $Q_{x^0}^0 \subset Q_{y^0}^0$, and Proposition 5.2 then tells us $x^0 = y^0$. As it is evident that we cannot have $Q_{x^0} \subset I^-((t, y^0, y'))$, that shows all we need.

To show the topology of \widehat{M} is that of \bar{M} first consider a sequence σ in $\hat{\partial}(M)$, *i.e.*, $\sigma(n) = Q_{x_n^0}$ for some sequence $\{x_n^0\}$ in K^0 . We want to show $Q_{x^0} \in L(\sigma)$ if and only if $[x^0] = \lim[x_n^0]$. Since the boundary is spacelike, we have $Q_{x^0} \in L(\sigma)$ if and only if for all $(t, y^0, y') \in Q_{x^0}$, eventually $(t, y^0, y') \in Q_{x_n^0}$. Since $Q_{x^0} = Q_{x^0}^0 \times K'$, this is equivalent to having for all $(t, y^0) \in Q_{x^0}^0$, eventually $(t, y^0) \in Q_{x_n^0}^0$. By Proposition 5.2, we know that is equivalent to $x^0 = \lim\{x_n^0\}$, and that is equivalent

to convergence in \bar{M} , i.e., $[x^0] = \lim\{[x_n^0]\}$. (Note that we don't have to worry about an element of M being in $L(\sigma)$, since $\hat{\partial}(M)$ is closed, M being a spacetime.)

The last thing to consider is a sequence σ in M , $\sigma(n) = (t_n, x_n^0, x'_n)$; we want to show $Q_{x^0} \in L(\sigma)$ is equivalent to $[x^0] = \lim\{\sigma(n)\}$, i.e., $\lim\{t_n\} = b$ and $\lim\{x_n^0\} = x^0$. We have $Q_{x^0} \in L(\sigma)$ if and only if for all $(t, y^0, y') \in Q_{x^0}$, eventually $(t, y^0, y') \ll (t_n, x_n^0, x'_n)$, and that is equivalent to having for all $(t, y^0) \in Q_{x^0}^0$ and all $y' \in K'$, eventually $(t, y^0, y') \ll (t_n, x_n^0, x'_n)$. In particular, this implies that for all $t < b$ and any $y' \in K'$, eventually $(t, x^0, y') \ll (t_n, x_n^0, x'_n)$, which implies eventually $(t, x^0) \ll (t_n, x_n^0)$ in M^0 ; and by Proposition 5.2, that implies $\lim\{t_n\} = b$ and $\lim\{x_n^0\} = x^0$. Conversely, suppose $\lim\{t_n\} = b$ and $\lim\{x_n^0\} = x^0$. By Proposition 5.2, we know $Q_{x^0}^0 \in L(\{(t_n, x_n^0)\})$, so for all $(t, y^0) \in Q_{x^0}^0$, eventually $(t, y^0) \ll (t_n, x_n^0)$ in M^0 . But by the analysis above for timelike relations in M , we know that for t fixed, if t_n is sufficiently close to b , then $(t, y^0) \ll (t_n, x_n^0)$ is sufficient to imply for any $z', w' \in K'$, $(t, y^0, z') \ll (t_n, x_n^0, w')$; and, in particular, that for any $y' \in K'$, $(t, y^0, y') \ll (t_n, x_n^0, x'_n)$. Since $\{t_n\}$ approaches b , we have this; therefore, $Q_{x^0} \in L(\{(t_n, x_n^0, x'_n)\})$. \square

As examples, consider the quasi-Kasner spacetimes, $K = (0, \infty) \times_{f_1} K_1 \times_{f_2} K_2 \times_{f_3} K_3$, where each K_i is one-dimensional—either \mathbb{R} or a circle—and $f_i(t) = t^{2p_i}$. We have $\int_1^\infty f_i^{-1/2} < \infty$ if and only if $p_i > 1$. Thus, if all $p_i \leq 1$ (as in the classical Kasner spacetimes, which also require the sums of $\{p_i\}$ and of $\{p_i^2\}$ each to be 1), then $\hat{\partial}(K)$ has null relations unless all three of the factors are circles, in which case the boundary is a single point. (This corrects a misstatement in section 5.2 of [H2].) If $p_1 > 1$ while $p_2 \leq 1$ and $p_3 \leq 1$, then $\hat{\partial}(K)$ has null relations unless K_2 and K_3 are circles; in that case, the boundary is K_1 . And so on for other combinations; we see that the boundary can be spacelike and any of a point, a line, a circle, a plane, a cylinder, a torus, \mathbb{R}^3 , a plane cross a circle, a line cross a torus, or a 3-torus, depending on the exponents and the spacelike factors.

With the structure of the boundary of multi-warped spacetimes in hand, we can see how they fit into Theorem 3.4. The interesting fact is that no extra assumption about \hat{V}/G need be made:

Proposition 3.6. *Let M be a multi-warped spacetime covered by a spacetime V supporting a free, properly discontinuous, chronological group action from a group G so that $M = V/G$. Suppose V and M have only spacelike future boundaries. Then \widehat{M} is homeomorphic \hat{V}/G , and $\hat{\partial}(M)$ is homeomorphic to $\hat{\partial}(V)/G$, i.e., $\widehat{V}/G \cong \hat{V}/G$ and $\hat{\partial}(V/G) \cong \hat{\partial}(V)/G$.*

Proof. We have $M = (a, b) \times_{f_1} K_1 \cdots \times_{f_m} K_m$. Arrange the factors so that for some k , $\int_{b-}^b f_i^{-1/2} < \infty$ for $i \leq k$ and $\int_{b-}^b f_i^{-1/2} = \infty$ for $i > k$; then, by Proposition 3.5, each K_i is complete, and for $i > k$, K_i is compact.

It is easy to work out what V is: Since V covers M , V is a quotient of the universal cover for M , \widetilde{M} ; in fact, G must be a normal subgroup of the fundamental group of M , $H = \pi_1(M)$, and $V = \widetilde{M}/(H/G)$. Nor is there any difficulty in determining what \widetilde{M} is: It must be $(a, b) \times_{f_1} \widetilde{K}_1 \cdots \times_{f_m} \widetilde{K}_m$ (with \widetilde{K}_i denoting the universal cover of K_i), as this is plainly simply connected and also a cover of M . Evidently, $H = H_1 \times \cdots \times H_m$, where $H_i = \pi_1(K_i)$ and H acts on \widetilde{M} via $(h_1, \dots, h_m) \cdot (t, x_1, \dots, x_m) = (t, h_1 \cdot x_1, \dots, h_m \cdot x_m)$. Since G is a subgroup of the product group H , G must also break down with factors $G = G_1 \times \cdots \times G_m$, each G_i

a subgroup of H_i ; furthermore, as G is normal in H , each G_i is normal in H_i , and $H/G \cong (H_1/G_1) \times \cdots \times (H_m/G_m)$. It follows that $V = (a, b) \times_{f_1} \bar{K}_1 \cdots \times_{f_m} \bar{K}_m$, where $\bar{K}_i = \tilde{K}_i/(H_i/G_i)$. Completeness and the integrals of the warping functions are unaffected by all these covers and quotients, but compactness may not be. For V to have a spacelike future boundary we must have \bar{K}_i compact for all $i > k$.

By Proposition 3.5, $\hat{\partial}(V) \cong \bar{K}_1 \times \cdots \times \bar{K}_k$. The action of G on $\hat{\partial}(V)$ derives from the action of H on \tilde{M} , so $(g_1, \dots, g_m) \cdot (x_1, \dots, x_k) = (g_1 \cdot x_1, \dots, g_k \cdot x_k)$ (with g_{k+1} through g_m irrelevant). This action is not free (if $k < m$), but we can analyze it in terms of $G^0 = G_1 \times \cdots \times G_k$, which does act freely on $\hat{\partial}(V)$, since G acts freely on V (as that implies each G_i acts freely on K_i). What is particularly important is that G^0 acts properly discontinuously on K^0 .

Now consider an element of $\hat{\partial}(V)/G^0$: It is an equivalence class $\langle Q_{x^0} \rangle$ for some $x^0 \in K^0 = K_1 \times \cdots \times K_k$, under the equivalence relation of movement by elements of G (using the Q -notation of the proof of Proposition 3.5). The past of $\langle Q_{x^0} \rangle$ in the chronological set \widehat{V}/G is $\{\langle p \rangle \mid p \in Q_{x^0}\}$, *i.e.*, the G -orbits of all points in Q_{x^0} . Suppose $\langle Q_{y^0} \rangle$ has the same past as $\langle Q_{x^0} \rangle$; then all G -orbits of points in Q_{x^0} are also G -orbits of points in Q_{y^0} , and *vice versa*. In particular, for all $t < b$, for any $x' \in K' = K_{k+1} \times \cdots \times K_m$, there is some $g \in G$ such that $g \cdot (t, x^0, x') \in Q_{y^0}$. This amounts to saying that for all n , there is some $g_n^0 \in G^0$ such that (for b finite) $(b - \frac{1}{n}, g_n^0 \cdot x^0) \in Q_{y^0}^0$ or (for $b = \infty$) $(n, g_n^0 \cdot x^0) \in Q_{y^0}^0$ (using the Q^0 -notation from the proof of Proposition 3.5). The proof of Proposition 5.2 shows that the s -slices—*i.e.*, intersections with $\{s\} \times K^0$ —of each $Q_{z^0}^0$ narrow down to $\{z^0\}$ as s approaches b (that is how it was shown that $Q_{z^0}^0 = Q_{w^0}^0$ implies $z^0 = w^0$). Thus we have $\{g_n^0 \cdot x^0\}$ approaches y^0 . Since G^0 acts properly discontinuously on K^0 , this can happen only if x^0 and y^0 are in the same G^0 -orbit. Therefore, $I^-(\langle Q_{x^0} \rangle) = I^-(\langle Q_{y^0} \rangle)$ implies $\langle x^0 \rangle = \langle y^0 \rangle$, which implies $\langle Q_{x^0} \rangle = \langle Q_{y^0} \rangle$: \widehat{V}/G is past-distinguishing. By Theorem 3.4, $\widehat{V}/G = \widehat{V}/G$ and $\hat{\partial}(V)/G = \hat{\partial}(V)/G$. \square

Proposition 3.6 shows that when a multi-warped spacetime is covered, via group action, by another spacetime, the covering spacetime is also multi-warped; and then the behavior of the boundaries, if spacelike, is very simple. But what if we start with the covering spacetime being multi-warped; does it follow that its quotient by a group action is also multi-warped? In other words: When examining $V \rightarrow M$, a principal covering projection, we now know that M being multi-warped propagates up the projection to the same being true for V ; but what about the converse, with V being assumed multi-warped?

If no assumptions about the spacelike nature of the boundaries are made, the group action on V can be more complicated than that established in Proposition 3.6. Consider, for instance, $V = \mathbb{L}^1 \times_f \mathbb{R}^1$, with $f(t)$ positive but bounded above (so V has a null boundary). We can act isometrically on V by the integers \mathbb{Z} via $m \cdot (t, x) = (t + \frac{1}{2}m, x + m)$, so long as f is periodic with period $\frac{1}{2}$; this will be a chronological action so long as $f(t)$ is sufficiently close to 1 for all t . Then V/\mathbb{Z} is a strongly causal spacetime but does not have the structure of a multi-warped spacetime unless f is very specially chosen: For a multi-warped spacetime will be of the form $\mathbb{L}^1 \times_\phi \mathbb{S}_D^1$ for some function ϕ with \mathbb{S}_D^1 the circle of diameter D . As $\mathbb{L}^1 \times_\phi \mathbb{S}_D^1$ contains a closed spacelike geodesic of length D , so must V/\mathbb{Z} if it is multi-warped, which translates into a spacelike geodesic in V which is preserved by the \mathbb{Z} -action. That means D must be the spacelike separation between $p = (0, 0)$

and $q = (\frac{1}{2}, 1)$, and a preserved geodesic must go between those points. But while there is surely a spacelike geodesic γ from p to q , only a special choice of f will allow for $\dot{\gamma}(D)$ to be the image of $\dot{\gamma}(0)$ by the translation $(t, x) \mapsto (t + \frac{1}{2}, x + 1)$.

Does imposing the restriction of a spacelike boundary alter things for the simpler? If the timelike factor for V is future-finite, *i.e.*, if the (a, b) factor has b finite, then the answer is yes: Any isometry of V must preserve the length of the longest timelike curve issuing from a point. From a point (t, x) , that length is $b - t$, so the isometry must preserve the $\{t\}$ -slice in V . Similarly, the length of the longest timelike curve between any two points must be preserved, so if the isometry takes (t, x) to (t, y) , then it must take (s, x) to (s, y) , as (s, y) is the only point in the $\{s\}$ -slice with longest curve-length $|t - s|$ from (t, y) . Thus, any group acting by isometries on V must act in the manner described in Proposition 3.6, and the quotient M is again multi-warped. Then Proposition 3.6 applies.

But what if V has spacelike boundary with $b = \infty$? That is unclear. No examples with irregular group action have come to light to date, but a proof that the group action must be as above seems elusive.

Example with $\hat{\pi}$ not continuous.

Here is an example demonstrating what can go wrong in the absence of spacelike boundaries.

Consider the following: Let M be the open subset of \mathbb{L}^2 formed by removing an infinite number of closed timelike line segments: Let L^- and L^+ be respectively the null lines $\{t = x\}$ and $\{t = x + 3\}$. Let σ^- and σ^+ be timelike curves respectively asymptotic to L^- and L^+ , $\sigma^-(s) = (\cosh s, \sinh s)$ and $\sigma^+(s) = (\cosh s, (\sinh s) + 3)$. For each positive integer n , let S_n be the closed segment from $\sigma^-(n)$ to $\sigma^+(n)$. Let $M = \mathbb{L}^2 - \bigcup \{S_n \mid n \geq 1\}$. Let $V = \widetilde{M}$, the universal cover of M , and let $G = \pi_1(M)$, the fundamental group of M acting on V so that $V/G = M$. Let $\pi : V \rightarrow M$ be the projection.

The elements of G are the homotopy classes of loops in M based at, say, $(0, 0)$. These can be realized as ordered n -tuples of integers, $s = (s_1, \dots, s_n)$, representing the sequence of slits around which the loop travels, with positive entries representing counterclockwise travel around the indicated slit (i th slit is S_k if $s_i = k > 0$) and negative numbers clockwise travel about the slit (S_k for $s_i = -k < 0$); the empty sequence is the identity element. We have the identifications $s \equiv s'$, in case $s_i = -s_{i+1}$ and s' is s with s_i and s_{i+1} deleted. The group operation is concatenation of sequences.

V can be realized as an infinite collection of identical sheets, parametrized by G , with gluings among them according to G . Specifically, let N be the manifold obtained from M by deleting horizontal slits running from each S_n to S_{n+1} : Let $H_n = \{(x, (\sinh n) + 2) \mid \cosh n < x < \cosh(n + 1)\}$; then $N = M - \bigcup \{H_n \mid n \geq 1\}$. Note that N is homeomorphic to the plane, since it is \mathbb{L}^2 with a connected closed tree-like set deleted. We realize V as $N \times G$ with the sheet $N \times \{s\}$ glued to the sheet $N \times \{s \cdot (k)\}$ across H_k (points on upper edge of H_k in $N \times \{s\}$ connected to points on lower edge of H_k in $N \times \{s \cdot (k)\}$).

Let P and Q be the IPs in M generated respectively by the future chains $c^+ = \{\sigma^+(n + 1/2)\}$ and $c^- = \{\sigma^-(n + 1/2)\}$. Actually, $P = I^-[L^+]$, but Q does not have so neat a formulation, as portions of the slits $\{S_n\}$ keep it from extending to L^- ; but the boundary of Q is asymptotic to L^- . (Note that P properly contains Q ; this is the crucial non-spacelike aspect of the boundary in M .) Let \tilde{c}^+ and

\tilde{c}^- be pre-images under π , respectively, of c^+ and c^- in $N \times \{e\}$ ($e = ()$, the identity element in G), and let $\tilde{P} = I^-[\tilde{c}^+]$ and $\tilde{Q} = I^-[\tilde{c}^-]$, IPs in V mapping via π onto P and Q respectively; \tilde{Q} maps homeomorphically to Q , but \tilde{P} is more complicated, as it extends across each H_k into $N \times \{(k)\}$, where it occupies a portion of the IP in $N \times \{(k)\}$ mapping homeomorphically onto Q —that portion with $t < -x + \cosh(k+1)$.

Just as \tilde{P} extends across H_k in $N \times \{e\}$ to a portion of the pre-image of Q lying in $N \times \{(k)\}$ (i.e., $(k) \cdot \tilde{Q}$), so $\tilde{P}_k = (-k) \cdot \tilde{P}$ extends across H_k in $N \times \{(-k)\}$ to a portion of \tilde{Q} —that portion with $t < -x + \cosh(k+1)$, points to the past of the null line through $\tilde{c}(k+1)$ rising to the left. Accordingly, every point in \tilde{Q} is contained in \tilde{P}_k for k sufficiently large. Furthermore, for every IP I properly containing \tilde{Q} , there is some point in I that fails to lie in \tilde{P}_k for k sufficiently large (because any IP properly containing \tilde{Q} must contain, for some $\delta > 0$, $\{(x, t) \in N \times \{e\} \mid t < x + \delta\}$). This says precisely that \tilde{Q} is a limit, in the $\hat{\cdot}$ -topology of \hat{V} , of the sequence $\{\tilde{P}_k \mid k \geq 1\}$.

We have $\hat{\pi}(\tilde{P}_k) = P$ for each k and $\hat{\pi}(\tilde{Q}) = Q$. Thus, if $\hat{\pi}$ were continuous, the constant sequence $\{P\}$ would have to have Q as a limit in the $\hat{\cdot}$ -topology of \hat{M} . But as M is past-distinguishing (being strongly causal), so is \hat{M} (by Theorem 5 of [H1]), so Proposition 2.1 of [H2] tells us that points in \hat{M} are closed; thus, the only limit of the constant sequence $\{P\}$ is P . (More concretely: P is an IP properly containing Q such that every element of P is eventually contained in the elements of the constant sequence $\{P\}$; thus Q is not a limit of that sequence.) Therefore, $\hat{\pi} : \hat{V} \rightarrow \hat{M}$ is not continuous.

(This is also an example of $\Pi : \hat{V} \rightarrow (\hat{V}/G, \hat{\cdot}\text{-topology})$ being not continuous: Since each of the \tilde{P}_k are G -related to \tilde{P} , we have for all k , $\Pi(\tilde{P}_k) = \langle \tilde{P}_k \rangle = \langle \tilde{P} \rangle$, so $\{\Pi(\tilde{P}_k)\}$ is the constant sequence $\{\langle \tilde{P} \rangle\}$. But just as P is an IP in M properly containing Q , so is \tilde{P} an IP in V properly containing \tilde{Q} , and so is $\langle \tilde{P} \rangle$ an IP in V/G properly containing $\langle \tilde{Q} \rangle$. Therefore, $\Pi(\tilde{Q}) = \langle \tilde{Q} \rangle \notin L(\{\Pi(\tilde{P}_k)\})$, even though $\tilde{Q} \in L(\{\tilde{P}_k\})$.)

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